Predictions of a fundamental statistical picture

Roland E. Allen

Department of Physics and Astronomy

Texas A&M University, College Station, Texas 77843, USA

Abstract

A picture is presented in which standard physics and its extensions are obtained from statistical counting and stochastic fluctuations, together with the geography of our particular universe in D dimensions. The inescapable predictions include supersymmetry, $SO(d)$ grand unification, Higgs-like bosons, vanishing of the usual cosmological constant, nonstandard behavior of scalar bosons, and Lorentz violation at extremely high energies.

I. INTRODUCTION

For a theory to be viable, it must be mathematically consistent, its premises must lead to testable predictions, and these predictions must be consistent with experiment and observation. Here we will present a theory which appears to satisfy these requirements, but which starts with an unfamiliar point of view: There are initially no laws, and instead all possibilities are realized with equal probability. The observed laws of Nature are emergent phenomena, which result from statistical counting and stochastic fluctuations, together with the geography (i.e. specific features) of our particular universe in D dimensions.

It is likely, of course, that experiment will soon confront theory with more stringent constraints. For example, supersymmetry [1–16], $SO(d)$ grand unification [1, 15–24], and Higgs-like bosons [1, 25–27] appear to be unavoidable consequences of the theory presented here, but there is as yet no direct evidence for any of these extensions of established physics.

Many of the ideas presented here appeared in a more fragmentary and preliminary way in an earlier paper [28] and conference proceedings [29]. The present paper is self-contained, however, and it supersedes all of these precursors.

II. STATISTICAL ORIGIN OF THE INITIAL ACTION

Our starting point is a single fundamental system which consists of identical (but distinguishable) irreducible objects, which we will call "whits" (because the alternative term "bit" has connotations which may be misleading). Each whit can exist in any of N_S states, with the number of whits in the ith state represented by n_i. An unobservable microstate of the fundamental system is specified by the number of whits and the state of each whit. An observable macrostate is specified by only the occupancies n_i of the states. As discussed below, D of the states are used to define D spacetime coordinates x^M, and N_F of the states are used to define fields ϕ_k. In this picture, Nature consists of all possible macrostates of the fundamental system, which means all possible combinations of n_k and n_M, which are equivalent to all possible values of the function $n_k(n_M)$, or all possible values of $\phi_k\left(x^M\right)$ up to phase factors (with the usual convention that x^M or n_M here represents the full set of D coordinates or occupancies).

Let us begin with the coordinates:

$$x^M = \pm n_M a_0 \tag{2.1}$$

where $M = 0, 1, ..., D-1$. It is convenient to include a fundamental length a_0 in this definition, so that we can later express the coordinates in conventional units. One can think of a_0 as being some multiple of the Planck length ℓ_P.

Notice that positive and negative coordinates correspond to the same occupancies, and that this definition is nevertheless physically acceptable: First, two points with coordinates that differ by minus signs are typically separated by cosmologically large distances in external spacetime. Second, and more importantly, the metric tensor and other physical fields defined below need not return to their original values when they are followed along a classical trajectory from a point with positive x^M to its counterpart with negative x^M. I. e., the geometry (and other field configurations) described by the two points can be regarded as distinct, and in this sense the points themselves are distinct. These different branches of the classical geometry (plus fields) are analogous to the branches of a multivalued function like $z^{1/2}$ or $\log z$, which are taken to correspond to different Riemann sheets, with the points on different sheets regarded as distinct.

Now define a set of fields ϕ_k by

$$\phi_k^2(x) = \rho_k(x) \quad , \quad k = 1, 2, ..., N_F \tag{2.2}$$

where

$$\rho_k(x) = n_k(x) / a_0^D \tag{2.3}$$

(and x represents all the coordinates). These initial bosonic fields ϕ_k are then real, and defined only up to a phase factor ± 1.

Let $S(x)$ be the entropy at a fixed point x, as defined by $S(x) = \log W(x)$ (in the units used throughout this paper, with $k_B = \hbar = c = 1$). Here $W(x)$ is the total number of microstates for fixed occupation numbers n_i: $W(x) = N(x)!/\Pi_i n_i(x)!$, with

$$N(x) = \sum_i n_i(x) \quad , \quad i = 1, 2, ..., N_S \; . \tag{2.4}$$

The total number of available microstates for all points x is $W = \Pi_x W(x)$, so the total entropy is

$$S = \sum_x S(x) \quad , \quad S(x) = \log \Gamma(N(x) + 1) - \sum_i \log \Gamma(n_i(x) + 1) \; . \tag{2.5}$$

We will see below that $n_k(x)$ can be approximately treated as a continuous variable when it is extremely large, with

$$\frac{\partial S}{\partial n_k(x)} = \psi\left(N(x)+1\right) - \psi\left(n_k(x)+1\right) \tag{2.6}$$

$$\frac{\partial^2 S}{\partial n_{k'}(x)\,\partial n_k(x)} = \psi^{(1)}\left(N(x)+1\right) - \psi^{(1)}\left(n_k(x)+1\right)\delta_{k'k}\ . \tag{2.7}$$

The functions $\psi(z) = d\log\Gamma(z)/dz$ and $\psi^{(1)}(z) = d^2\log\Gamma(z)/dz^2$ have the asymptotic expansions [30]

$$\psi(z) = \log z - \frac{1}{2z} - \sum_{l=1}^{\infty}\frac{B_{2l}}{2l\,z^{2l}} \quad , \quad \psi^{(1)}(z) = \frac{1}{z} + \frac{1}{2z^2} + \sum_{l=1}^{\infty}\frac{B_{2l}}{z^{2l+1}} \tag{2.8}$$

as $z\to\infty$. It will be assumed that each $n_k(x)$ has some characteristic value $\overline{n}_k(x)$ which is vastly larger than nearby values:

$$n_k(x) = \overline{n}_k(x) + \Delta n_k(x) \quad , \quad \overline{n}_k(x) >>> |\Delta n_k(x)| \tag{2.9}$$

where ">>>" means "is vastly greater than", as in $10^{200} >>> 1$. This assumption is consistent with the fact that the initial action of (2.27) and (2.28) has no lower bound, so that $n_k(x)\to\infty$ before the extra stochastic term involving (2.31) is added. Then it is an extremely good approximation to use the asymptotic formulas above and write

$$S = S_0 + \sum_{x,k} a_k(x)\,\Delta n_k(x) - \sum_{x,k} a'_k(x)\left[\Delta n_k(x)\right]^2 + \sum_{x,k,k'\neq k} a'_{kk'}(x)\,\Delta n_k(x)\,\Delta n_{k'}(x) \tag{2.10}$$

$$a_k(x) = \log\overline{N}(x) - \log\overline{n}_k(x) \tag{2.11}$$

$$a'_k(x) = \left(2\overline{n}_k(x)\right)^{-1} - \left(2\overline{N}(x)\right)^{-1} \quad , \quad a'_{kk'}(x) = \left(2\overline{N}(x)\right)^{-1} \tag{2.12}$$

where $\overline{N}(x)$ is the value of $N(x)$ when $n_k(x) = \overline{n}_k(x)$ for all k, and the higher-order terms have been separately neglected in $a_k(x)$ and $a'_k(x)$. For simplicity, we will also neglect the terms involving $\left(2\overline{N}(x)\right)^{-1}$. Since there is initially no distinction between the fields labeled by k, it is reasonable to assume that they all have the same $\overline{n}_k(x) = \overline{n}(x)$. It is also reasonable to assume that $\overline{n}(x)$ is nearly independent of x (except possibly in extreme situations like the very early universe): $\overline{n}(x) = \overline{n}$ and $\overline{N}(x) = \overline{N}$, so that

$$a_k(x) = a = \log\left(\overline{N}/\overline{n}\right) \tag{2.13}$$

$$a'_k(x) = a' = (2\overline{n})^{-1}\ . \tag{2.14}$$

It is not conventional or convenient to deal with $\Delta n_k(x)$ and $[\Delta n_k(x)]^2$, so let us instead write S in terms of the fields ϕ_k and their derivatives $\partial\phi_k/\partial x^M$ via the following procedure: First, we can switch to a new set of points $\overline{x}$, defined to be the corners of the D-dimensional hypercubes centered on the original points x. It is easy to see that

$$S = S_0 + \sum_{\overline{x},k} a \langle \Delta n_k(x) \rangle - \sum_{\overline{x},k} a' \left\langle [\Delta n_k(x)]^2 \right\rangle \tag{2.15}$$

where $\langle \cdots \rangle$ in the present context indicates an average over the 2^D boxes labeled by x which have the common corner $\overline{x}$. Second, at each point x we can write $\Delta n_k = \Delta\rho_k a_0^D = (\langle \Delta\rho_k \rangle + \delta\rho_k) a_0^D$, with $\langle \delta\rho_k \rangle = 0$:

$$\begin{aligned} S &= S_0 + \sum_{\overline{x},k} a \langle \langle \Delta\rho_k \rangle + \delta\rho_k \rangle a_0^D - \sum_{\overline{x},k} a' \left\langle (\langle \Delta\rho_k \rangle + \delta\rho_k)^2 \right\rangle \left(a_0^D\right)^2 && (2.16) \\ &= S_0 + \sum_{\overline{x},k} a \langle \Delta\rho_k \rangle a_0^D - \sum_{\overline{x},k} a' \left[\langle \Delta\rho_k \rangle^2 + \left\langle (\delta\rho_k)^2 \right\rangle \right] \left(a_0^D\right)^2 \ . && (2.17) \end{aligned}$$

Each of the points x surrounding $\overline{x}$ is displaced by $\delta x^M = \pm a_0/2$ along each of the x^M axes, so

$$\begin{aligned} \left\langle (\delta\rho_k)^2 \right\rangle &= \left\langle \left(\delta\phi_k^2\right)^2 \right\rangle && (2.18) \\ &= \left\langle \sum_M \left(\frac{\partial\phi_k^2}{\partial x^M} \delta x^M + \frac{1}{2} \frac{\partial^2 \phi_k^2}{\partial (x^M)^2} \left(\delta x^M\right)^2 \right)^2 \right\rangle && (2.19) \\ &= \left\langle \sum_M \left(2\phi_k \frac{\partial\phi_k}{\partial x^M} \delta x^M + \left(\frac{\partial\phi_k}{\partial x^M} \right)^2 \left(\delta x^M\right)^2 + \phi_k \frac{\partial^2 \phi_k}{\partial (x^M)^2} \left(\delta x^M\right)^2 \right)^2 \right\rangle && (2.20) \end{aligned}$$

to lowest order, where it is now assumed that at normal energies the fields are slowly varying over the extremely small distance a_0. This assumption is justified by the prior assumption that $\overline{n}$ is extremely large: $\phi_k^2(x) = \rho_k(x) = n_k(x)/a_0^D$ implies that $2\delta\phi_k/\phi_k \approx \delta n_k/n_k$ and $\phi_k = n_k^{1/2} a_0^{-D/2}$, so that $\delta\phi_k \sim \delta n_k \, n_k^{-1/2} a_0^{-D/2}$. The minimum change in ϕ_k is given by $\delta n_k = 1 : \delta\phi_k^{\min} \sim n_k^{-1/2} a_0^{-D/2}$, which means that $\delta\phi_k^{\min}$ is extremely small if n_k is extremely large. For example, $D = 4$, $a_0^{-1} \sim 10^{15}$ TeV, and $n_k \sim 10^{200}$ would give $\delta\phi_k^{\min} \sim 10^{-70}$ TeV$^2 \sim 10^{-38}$ K^2.

In other words, the fields ϕ_k have effectively continuous values as $\overline{n} \to \infty$, just as the spacetime coordinates have effectively continuous values as $a_0 \to 0$.

For extremely large $\overline{n}$ it is an extremely good approximation to neglect the middle term in (2.20), and to replace ϕ_k^2 by

$$\overline{\phi}^2 = \overline{\rho} = \overline{n}/a_0^D \tag{2.21}$$

giving

$$a' \left\langle (\delta\rho_k)^2 \right\rangle = \frac{1}{2a_0^D} \sum_M \left[\left(\frac{\partial \phi_k}{\partial x^M} \right)^2 a_0^2 + \left(\frac{\partial^2 \phi_k}{\partial \left(x^M\right)^2} \right)^2 \frac{a_0^4}{16} \right] . \tag{2.22}$$

It is similarly an extremely good approximation to neglect the term in (2.17) involving $a' \left(a_0^D\right)^2 \langle \Delta\rho_k \rangle^2 = \langle \Delta n_k \rangle^2 / 2\overline{n}$ in comparison to that involving $\langle \Delta\rho_k \rangle a_0^D = \langle \Delta n_k \rangle$, so that

$$S = S_0 + \sum_{\overline{x},k} a_0^D \frac{\mu_0}{m_0} \left(\phi_k^2 - \overline{\phi}^2 \right) - \sum_{\overline{x},k} \sum_M a_0^D \frac{1}{2m_0^2} \left[\left(\frac{\partial \phi_k}{\partial x^M} \right)^2 + \frac{a_0^2}{16} \left(\frac{\partial^2 \phi_k}{\partial \left(x^M\right)^2} \right)^2 \right] \tag{2.23}$$

where

$$m_0 = a_0^{-1} \quad , \quad \mu_0 = m_0 \log \left(\overline{N} / \overline{n} \right) . \tag{2.24}$$

The philosophy behind the above treatment is simple: We essentially wish to replace $\langle f^2 \rangle$ by $(\partial f / \partial x)^2$, and this can be accomplished because

$$\left\langle f^2 \right\rangle - \langle f \rangle^2 = \left\langle (\delta f)^2 \right\rangle \approx \left\langle (\partial f / \partial x)^2 (\delta x)^2 \right\rangle = (\partial f / \partial x)^2 (a_0/2)^2 . \tag{2.25}$$

The form of (2.23) also has a simple interpretation: The entropy S increases with the number of whits, but decreases when the whits are not uniformly distributed.

In the continuum limit,

$$\sum_{\overline{x}} a_0^D \rightarrow \int d^D x \tag{2.26}$$

(2.23) becomes

$$S = S_0 + \int d^D x \sum_k \left\{ \frac{\mu_0}{m_0} \left(\phi_k^2 - \overline{\phi}^2 \right) - \frac{1}{2m_0^2} \sum_M \left[\left(\frac{\partial \phi_k}{\partial x^M} \right)^2 + \frac{a_0^2}{16} \left(\frac{\partial^2 \phi_k}{\partial \left(x^M\right)^2} \right)^2 \right] \right\} . \tag{2.27}$$

A physical configuration of all the fields $\phi_k(x)$ corresponds to a specification of all the density variations $\Delta\rho_k(x)$. In the present picture, the probability of such a configuration is proportional to $W = e^S$. In a Euclidean path integral, the probability is proportional to $e^{-\overline{S}_b}$, where $\overline{S}_b$ is the Euclidean action for these bosonic fields. We conclude that

$$\overline{S}_b = -S + \text{constant} \tag{2.28}$$

and we will choose the constant to equal S_0.

In the following it will be convenient to write the action in terms of $\widetilde{\phi}_k = m_0^{-1/2} \phi_k$. For simplicity, we assume that the number of relevant $\widetilde{\phi}_k$ is even, so that we can group these

real fields in pairs to form N_f complex fields $\widetilde{\Psi}_{b,k}$. It is also convenient to subtract out the enormous contribution of $\overline{\phi}$ by defining

$$\Psi_b = \widetilde{\Psi}_b - \overline{\Psi}_b \tag{2.29}$$

where $\widetilde{\Psi}_b$ is the vector with components $\widetilde{\Psi}_{b,k}$ and $\overline{\Psi}_b$ is similarly defined with $\phi_k \to \overline{\phi}$. Then the action can be written

$$\overline{S}_b = \int d^D x \left\{ \frac{1}{2m_0} \left[\frac{\partial \Psi_b^\dagger}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^\dagger}{\partial \left(x^M\right)^2} \frac{\partial^2 \Psi_b}{\partial \left(x^M\right)^2} \right] - \mu_0 \left(\widetilde{\Psi}_b^\dagger \widetilde{\Psi}_b - \overline{\Psi}_b^\dagger \overline{\Psi}_b \right) \right\} \tag{2.30}$$

since $\overline{\Psi}_b$ is constant, with summation now implied over repeated indices like M.

At this point it is necessary to introduce another fundamental assumption, without which the action would have no lower bound: We assume that the whits are randomly perturbed by an unspecified environment whose effect can be modeled by an extra term $i\widetilde{V}\,\Psi_b^\dagger \Psi_b$. Here $\widetilde{V}$ is a potential which has a Gaussian distribution, with

$$\left\langle \widetilde{V} \right\rangle = 0 \quad , \quad \left\langle \widetilde{V}\left(x\right) \widetilde{V}\left(x'\right) \right\rangle = b\, \delta\left(x - x'\right) \tag{2.31}$$

where b is a constant. Then the complete action has the form

$$\begin{aligned} \widetilde{S}_B \left[\Psi_b^\dagger, \Psi_b \right] = \int d^D x \Bigg\{ & \frac{1}{2m_0} \left[\frac{\partial \Psi_b^\dagger}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^\dagger}{\partial \left(x^M\right)^2} \frac{\partial^2 \Psi_b}{\partial \left(x^M\right)^2} \right] \\ & - \mu_0 \left(\widetilde{\Psi}_b^\dagger \widetilde{\Psi}_b - \overline{\Psi}_b^\dagger \overline{\Psi}_b \right) + i\widetilde{V}\,\Psi_b^\dagger \Psi_b \Bigg\} . \end{aligned} \tag{2.32}$$

In the following we will assume that the only fields which make an appreciable contribution in (2.32) are those for which $\int d^D x\, \overline{\Psi}_b^\dagger \Psi_b = 0$. This assumption is consistent with the fact that $\overline{\Psi}_b$ is constant with respect to all the coordinates and, in the present picture, fields Ψ_b corresponding to physical gauge representations have nonzero angular momenta in the internal space of Section V and Appendices A and B. Then (2.32) simplifies to

$$\widetilde{S}_B \left[\Psi_b^\dagger, \Psi_b \right] = \int d^D x \left\{ \frac{1}{2m_0} \left[\frac{\partial \Psi_b^\dagger}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^\dagger}{\partial \left(x^M\right)^2} \frac{\partial^2 \Psi_b}{\partial \left(x^M\right)^2} \right] - \mu_0\, \Psi_b^\dagger \Psi_b + i\widetilde{V}\,\Psi_b^\dagger \Psi_b \right\} . \tag{2.33}$$

III. PRIMITIVE SUPERSYMMETRY

If F is any functional of the fundamental fields Ψ_b, its average value is given by

$$\left\langle F \right\rangle = \left\langle \frac{\int \mathcal{D}\,\Psi_b^\dagger\, \mathcal{D}\,\Psi_b\, F\left[\Psi_b^\dagger, \Psi_b\right]\; e^{-\widetilde{S}_B\left[\Psi_b^\dagger, \Psi_b\right]}}{\int \mathcal{D}\,\underline{\Psi}_b^\dagger\, \mathcal{D}\,\underline{\Psi}_b\, e^{-\widetilde{S}_B\left[\underline{\Psi}_b^\dagger, \underline{\Psi}_b\right]}} \right\rangle \tag{3.1}$$

where $\langle \cdots \rangle$ now represents an average over the perturbing potential $i\widetilde{V}$ and $\int \mathcal{D}\,\Psi_b^\dagger\,\mathcal{D}\,\Psi_b$ is to be interpreted as $\prod_{x,k} \int_{-\infty}^{\infty} d\,\mathrm{Re}\,\Psi_{b,k}\left(x\right) \int_{-\infty}^{\infty} d\,\mathrm{Im}\,\Psi_{b,k}\left(x\right)$. The transition from the original summation over $n_k\left(x\right)$ to this Euclidean path integral has the form (with $\Delta n = 1$ here)

$$\sum_{n=0}^{\infty} f\left(n\right)\Delta n \rightarrow \int_0^\infty f dn \rightarrow \int_0^\infty f\; d\left(a_0^D \phi^2\right) \rightarrow 2\overline{\phi} a_0^D \int_0^\infty f\; d\phi \rightarrow 2\overline{\phi}\, a_0^D m_0^{1/2} \int_{-\infty}^\infty f\; d\phi' \tag{3.2}$$

where $\phi' = \widetilde{\phi} - m_0^{-1/2}\overline{\phi}$, since $d\left(\phi^2\right) \approx 2\overline{\phi}\, d\phi$ is an extremely good approximation for physically relevant fields, and since ϕ' effectively ranges from $-\infty$ to $+\infty$. Each ϕ' then becomes a $\mathrm{Re}\,\Psi_{b,k}\left(x\right)$ or $\mathrm{Im}\,\Psi_{b,k}\left(x\right)$, and the constant factors cancel in the numerator and denominator of (3.1).

The presence of the denominator makes it difficult to perform the average of (3.1), but there is a trick for removing the bosonic degrees of freedom $\underline{\Psi}_b$ in the denominator and replacing them with fermionic degrees of freedom Ψ_f in the numerator [31–33]: After integration by parts, (2.33) can be written in the form $\widetilde{S}_B = \int d^D x\, \Psi_b^\dagger A \Psi_b$. Then, since

$$\int \mathcal{D}\,\underline{\Psi}_b^\dagger\,\mathcal{D}\,\underline{\Psi}_b\, e^{-\widetilde{S}_B\left[\underline{\Psi}_b^\dagger, \underline{\Psi}_b\right]} = C\left(\det \mathcal{A}\right)^{-1} \tag{3.3}$$

$$\int \mathcal{D}\,\Psi_f^\dagger\,\mathcal{D}\,\Psi_f\, e^{-\widetilde{S}_B\left[\Psi_f^\dagger, \Psi_f\right]} = \det \mathcal{A} \tag{3.4}$$

where the matrix $\mathcal{A}$ corresponds to the operator A and C is a constant, it follows that

$$\langle F \rangle = \frac{1}{C}\left\langle \int \mathcal{D}\,\Psi_b^\dagger\,\mathcal{D}\,\Psi_b\,\mathcal{D}\,\Psi_f^\dagger\,\mathcal{D}\,\Psi_f\, F\, e^{-\widetilde{S}_B\left[\Psi_b^\dagger,\Psi_b\right]} e^{-\widetilde{S}_B\left[\Psi_f^\dagger,\Psi_f\right]} \right\rangle \tag{3.5}$$

$$= \frac{1}{C}\left\langle \int \mathcal{D}\,\Psi^\dagger\,\mathcal{D}\,\Psi\, F\, e^{-\widetilde{S}_{bf}\left[\Psi^\dagger,\Psi\right]} \right\rangle \tag{3.6}$$

where Ψ_b and Ψ_f have been combined into

$$\Psi = \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix} \tag{3.7}$$

and

$$\widetilde{S}_{bf}\left[\Psi^\dagger, \Psi\right] = \int d^D x \left\{ \frac{1}{2m_0}\left[\frac{\partial \Psi^\dagger}{\partial x^M}\frac{\partial \Psi}{\partial x^M} + \frac{a_0^2}{16}\frac{\partial^2 \Psi^\dagger}{\partial \left(x^M\right)^2}\frac{\partial^2 \Psi}{\partial \left(x^M\right)^2} \right] - \mu_0\, \Psi^\dagger \Psi + i\widetilde{V}\, \Psi^\dagger \Psi \right\} . \tag{3.8}$$

In (3.7), Ψ_f consists of Grassmann variables $\Psi_{f,k}$, just as Ψ_b consists of ordinary variables $\Psi_{b,k}$, and $\int \mathcal{D}\,\Psi^\dagger\,\mathcal{D}\,\Psi$ is to be interpreted as

$$\prod\nolimits_{x,k} \int_{-\infty}^{\infty} d\,\mathrm{Re}\,\Psi_{b,k}\left(x\right) \int_{-\infty}^{\infty} d\,\mathrm{Im}\,\Psi_{b,k} \int d\,\Psi_{f,k}^*\left(x\right) \int d\,\Psi_{f,k}\left(x\right) . \tag{3.9}$$

Recall that Ψ_b and Ψ_f each have N_f components.

For a Gaussian random variable v whose mean is zero, the result

$$\left\langle e^{-iv} \right\rangle = e^{-\frac{1}{2}\left\langle v^2 \right\rangle} \tag{3.10}$$

implies that

$$\left\langle e^{-\int d^D x \, i\widetilde{V}\, \Psi^\dagger \Psi} \right\rangle = e^{-\frac{1}{2}\int d^D x \;\, d^D x' \, \Psi^\dagger(x)\Psi(x) \left\langle \widetilde{V}(x)\widetilde{V}(x') \right\rangle \Psi^\dagger(x')\Psi(x')} \tag{3.11}$$

$$= e^{-\frac{1}{2} b \int d^D x \;\, \left[\Psi^\dagger(x)\Psi(x)\right]^2} . \tag{3.12}$$

It follows that

$$\langle F \rangle = \frac{1}{C} \int \mathcal{D}\,\Psi^\dagger\, \mathcal{D}\,\Psi\, F\, e^{-S_E} \tag{3.13}$$

with

$$S_E = \int d^D x \left\{ \frac{1}{2m_0} \left[\frac{\partial \Psi^\dagger}{\partial x^M} \frac{\partial \Psi}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi^\dagger}{\partial \left(x^M\right)^2} \frac{\partial^2 \Psi}{\partial \left(x^M\right)^2} \right] - \mu_0\, \Psi^\dagger \Psi + \frac{1}{2} b \left(\Psi^\dagger \Psi\right)^2 \right\} . \tag{3.14}$$

A special case is

$$Z = \frac{1}{C} \int \mathcal{D}\,\Psi^\dagger\, \mathcal{D}\,\Psi e^{-S_E} \tag{3.15}$$

but according to (3.1) $Z = 1$, so $C = \int \mathcal{D}\,\Psi^\dagger\, \mathcal{D}\,\Psi e^{-S_E}$ and

$$\langle F \rangle = \frac{\int \mathcal{D}\,\Psi^\dagger\, \mathcal{D}\,\Psi\, F\, e^{-S_E}}{\int \mathcal{D}\,\Psi^\dagger\, \mathcal{D}\,\Psi\; e^{-S_E}} . \tag{3.16}$$

Notice that the fermionic variables Ψ_f represent true degrees of freedom, and that they originate from the bosonic variables $\underline{\Psi}_b$. The coupling between the fields Ψ_b and Ψ_f (or $\underline{\Psi}_b$) is due to the random perturbing potential $i\widetilde{V}$. In the replacement of (3.1) by (3.16), F essentially serves as a test functional. The meaning of this replacement is that the action (3.14), with both bosonic and fermionic fields, must be used instead of the original action (2.33), with only bosonic fields, in treating all physical quantities and processes, if the average over random fluctuations in (3.1) is to disappear from the theory.

Ordinarily we can let $a_0 \to 0$ in (3.14), so that

$$S_E = \int d^D x \left[\frac{1}{2m_0} \partial_M \Psi^\dagger \partial_M \Psi - \mu_0\, \Psi^\dagger \Psi + \frac{1}{2} b \left(\Psi^\dagger \Psi\right)^2 \right] . \tag{3.17}$$

However, the higher-derivative term in the action may be relevant in the internal space defined below, where the length scales can be comparable to a_0. Also, one should remember that the discreteness of spacetime in the present theory implies that a_0 automatically provides an ultimate ultraviolet cutoff.

IV. LORENTZ INVARIANCE

As mentioned in the first section, within the present theory the laws of Nature arise from (1) the statistical counting which leads to (2.30), (2) the stochastic fluctuations of (2.31), and (3) the geography (or specific features) of our universe, to which we now turn.

The most central assumption is that

$$\Psi_b = \Psi_0 + \Psi_b' \tag{4.1}$$

where Ψ_0 is the order parameter for a primordial bosonic condensate which forms in the very early universe, and Ψ_b' represents all the other bosonic fields. The treatment of Appendix A implies that

$$\Psi_0^\dagger \Psi_b' = 0 \tag{4.2}$$

everywhere. (See (A11) and the comments above (A2) and (A9).) The action can then be written as

$$\begin{aligned}
S_E &= S_{cond} + S_b + S_f + S_{int} && (4.3)\\
S_{cond} &= \int d^D x \left[\frac{1}{2m_0} \partial_M \Psi_0^\dagger \partial_M \Psi_0 - \mu_0 \Psi_0^\dagger \Psi_0 + \frac{1}{2} b \left(\Psi_0^\dagger \Psi_0 \right)^2 \right] && (4.4)\\
S_b &= \int d^D x \left[\frac{1}{2m_0} \partial_M \Psi_b'^\dagger \partial_M \Psi_b' + (V_0 - \mu_0)\ \Psi_b'^\dagger \Psi_b' + \frac{1}{2} b \left(\Psi_b'^\dagger \Psi_b' \right)^2 \right] && (4.5)\\
S_f &= \int d^D x \left[\frac{1}{2m_0} \partial_M \Psi_f^\dagger \partial_M \Psi_f + (V_0 - \mu_0)\ \Psi_f^\dagger \Psi_f + \frac{1}{2} b \left(\Psi_f^\dagger \Psi_f \right)^2 \right] && (4.6)\\
S_{int} &= \int d^D x \, b \left(\Psi_f^\dagger \Psi_f \right) \left(\Psi_b'^\dagger \Psi_b' \right) && (4.7)\\
V_0 &= b \, \Psi_0^\dagger \Psi_0 \, . && (4.8)
\end{aligned}$$

In the following, the last term will be neglected in (4.5) and (4.6); we are thus considering the theory prior to formation of further condensates beyond the primordial Ψ_0.

For a static condensate we could write $\Psi_0 = n_0^{1/2} \eta_0$, where η_0 is constant, $\eta_0^\dagger \eta_0 = 1$, and $n_0 = \Psi_0^\dagger \Psi_0$ is the condensate density. This picture is too simplistic, however, since the order parameter can exhibit rotations that are analogous to the rotations in the complex plane of the order parameter $\psi_s = e^{i\theta_s} n_s^{1/2}$ for an ordinary superfluid:

$$\Psi_0(x) = U_0(x)\ n_0(x)^{1/2}\, \eta_0 \quad , \quad U_0^\dagger U_0 = 1 \, . \tag{4.9}$$

After an integration by parts in (4.4) (with boundary terms always neglected in the present paper), extremalization of the action gives the classical equation of motion for the order parameter:

$$-\frac{1}{2m_0}\partial_M\partial_M\Psi_0+(V_0-\mu_0)\,\Psi_0=0\ . \tag{4.10}$$

(In this equation the ordinary field densities $\Psi_b'^\dagger\Psi_b'$ and $\Psi_f^\dagger\Psi_f$ are neglected in comparison to the primordial condensate density $\Psi_0^\dagger\Psi_0$.)

In specifying the geography of our universe, it will be further assumed that Ψ_0 can be written as the product of a 2-component external order parameter Ψ_{ext}, which is a function of 4 external coordinates x^μ, and an internal order parameter Ψ_{int}, which is primarily a function of d internal coordinates x^m, but which also varies with x^μ:

$$\Psi_0 \;=\; \Psi_{ext}\left(x^\mu\right)\,\Psi_{int}\left(x^m,x^\mu\right) \tag{4.11}$$

$$\Psi_{ext}\left(x^\mu\right) \;=\; U_{ext}\left(x^\mu\right)\,n_{ext}\left(x^\mu\right)^{1/2}\eta_{ext}\quad,\quad \mu=0,1,2,3 \tag{4.12}$$

$$\Psi_{int}\left(x^m,x^\mu\right) \;=\; U_{int}\left(x^m,x^\mu\right)\,n_{int}\left(x^m,x^\mu\right)^{1/2}\eta_{int}\quad,\quad m=4,...,D-1 \tag{4.13}$$

where again η_{ext} and η_{int} are constant, and $\eta_{ext}^\dagger\eta_{ext}=\eta_{int}^\dagger\eta_{int}=1$. Here, according to a standard notation, x^μ actually represents the set of x^μ, and x^m the set of x^m.

Let us define external and internal "superfluid velocities" by

$$m_0v_\mu=-iU_{ext}^{-1}\partial_\mu U_{ext}\quad,\quad m_0v_m=-iU_{int}^{-1}\partial_m U_{int}\ . \tag{4.14}$$

The fact that $U_{ext}^\dagger U_{ext}=1$ implies that $\left(\partial_\mu U_{ext}^\dagger\right)U_{ext}=-U_{ext}^\dagger\left(\partial_\mu U_{ext}\right)$ with $U_{ext}^\dagger=U_{ext}^{-1}$, or $m_0v_\mu=i\left(\partial_\mu U_{ext}^\dagger\right)U_{ext}$, so that

$$v_\mu^\dagger=v_\mu\ . \tag{4.15}$$

For simplicity, let us first consider the case

$$\partial_\mu U_{int}=0 \tag{4.16}$$

for which there are separate external and internal equations of motion:

$$\left(-\frac{1}{2m_0}\partial_\mu\partial_\mu-\mu_{ext}\right)\Psi_{ext}=0\quad,\quad\left(-\frac{1}{2m_0}\partial_m\partial_m-\mu_{int}+V_0\right)\Psi_{int}=0 \tag{4.17}$$

with

$$\mu_{int}=\mu_0-\mu_{ext}\ . \tag{4.18}$$

The quantities μ_{int} and V_0 have a relatively slow parametric dependence on x^μ.

When (4.12) and (4.14) are used in (4.17), we obtain

$$\eta_{ext}^{\dagger}\, n_{ext}^{1/2} \left[\left(\frac{1}{2} m_0 v_\mu v_\mu - \frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{ext} \right) - i \left(\frac{1}{2} \partial_\mu v_\mu + v_\mu \partial_\mu \right) \right] n_{ext}^{1/2} \eta_{ext} = 0 \tag{4.19}$$

and its Hermitian conjugate

$$\eta_{ext}^{\dagger}\, n_{ext}^{1/2} \left[\left(\frac{1}{2} m_0 v_\mu v_\mu - \frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{ext} \right) + i \left(\frac{1}{2} \partial_\mu v_\mu + v_\mu \partial_\mu \right) \right] n_{ext}^{1/2} \eta_{ext} = 0 \; . \tag{4.20}$$

Subtraction gives the equation of continuity

$$\partial_\mu\, j_\mu^{ext} = 0 \quad , \quad j_\mu^{ext} = n_{ext}\, \eta_{ext}^{\dagger}\, v_\mu \eta_{ext} \tag{4.21}$$

and addition gives the Bernoulli equation

$$\frac{1}{2} m_0 \bar{v}_{ext}^2 + P_{ext} = \mu_{ext} \tag{4.22}$$

where

$$\bar{v}_{ext}^2 = \eta_{ext}^{\dagger}\, v_\mu v_\mu\, \eta_{ext} \quad , \quad P_{ext} = -\frac{1}{2m_0} n_{ext}^{-1/2} \partial_\mu \partial_\mu n_{ext}^{1/2} \; . \tag{4.23}$$

Since the order parameter Ψ_{ext} in external spacetime has 2 components, its "superfluid velocity" v_μ can be written in terms of the identity matrix σ^0 and Pauli matrices σ^a :

$$v_\mu = v_\mu^\alpha \sigma^\alpha \quad , \quad \mu, \alpha = 0, 1, 2, 3 \; . \tag{4.24}$$

It will be assumed that the coordinate system can be chosen such that

$$v_k^0 = v_0^a = 0 \quad , \quad k, a = 1, 2, 3 \tag{4.25}$$

to a good approximation, yielding the simplification

$$\frac{1}{2} m_0 v_\mu^\alpha v_\mu^\alpha + P_{ext} = \mu_{ext} \; . \tag{4.26}$$

As v_μ^α varies, μ_{ext} varies in response, with μ_{int} determined by (4.18).

Now expand Ψ_b' in terms of a complete set of basis functions ψ_r^{int} in the internal space:

$$\Psi_b' \left(x^\mu, x^m \right) = \widetilde{\psi}_b^r \left(x^\mu \right) \widetilde{\psi}_{int}^r \left(x^m \right) \tag{4.27}$$

with

$$\left(-\frac{1}{2m_0} \partial_m \partial_m - \mu_{int} + V_0 \right) \widetilde{\psi}_{int}^r \left(x^m \right) \; = \; \varepsilon_r \widetilde{\psi}_{int}^r \left(x^m \right) \tag{4.28}$$

$$\int d^{D-4}x\, \widetilde{\psi}_{int}^{r\dagger} \left(x^m \right) \widetilde{\psi}_{int}^{r'} \left(x^m \right) \; = \; \delta_{rr'} \tag{4.29}$$

and with the usual summation over repeated indices. For reasons that will become fully apparent below, but which are already suggested by the form of the order parameter, each $\widetilde{\psi}_b^r(x^\mu)$ has two components. As usual, only the zero ($\varepsilon_r = 0$) modes will be kept. (To simplify the presentation, the higher-derivative terms are not explicitly shown in the present section; they will be restored in the next section.) When (4.27)-(4.29) are then used in (4.5) (with the last term neglected), the result is

$$S_b = \int d^4x\, \widetilde{\psi}_b^\dagger \left(-\frac{1}{2m_0}\partial_\mu\partial_\mu - \mu_{ext} \right) \widetilde{\psi}_b \tag{4.30}$$

where $\widetilde{\psi}_b$ is the vector with components $\widetilde{\psi}_b^r$.

Let $\widetilde{\psi}_b$ be written in the form

$$\widetilde{\psi}_b(x^\mu) = U_{ext}(x^\mu)\, \psi_b(x^\mu) \tag{4.31}$$

or equivalently

$$\widetilde{\psi}_b^r(x^\mu) = U_{ext}(x^\mu)\, \psi_b^r(x^\mu)\ . \tag{4.32}$$

Here ψ_b has a simple interpretation: It is the field seen by an observer in the frame of reference that is moving with the condensate. In the present theory, a (very high density) condensate Ψ_0 forms in the very early universe, and the other bosonic and fermionic fields are subsequently born into it. It is therefore natural to define the fields ψ_b^r in the condensate's frame of reference.

Equation (4.31) is, in fact, exactly analogous to rewriting the wavefunction of a particle in an ordinary superfluid moving with velocity v_s: $\widetilde{\psi}_{par}(x) = \exp(imv_s x)\, \psi_{par}(x)$. Here ψ_{par} is the wavefunction in the superfluid's frame of reference.

When (4.31) is substituted into (4.30), the result is

$$S_b = \int d^4x\ \psi_b^\dagger \left[\left(\frac{1}{2}m_0 v_\mu v_\mu - \frac{1}{2m_0}\partial_\mu\partial_\mu - \mu_{ext} \right) - i\left(\frac{1}{2}\partial_\mu v_\mu + v_\mu\partial_\mu \right) \right] \psi_b\ . \tag{4.33}$$

If n_s and v_μ are slowly varying, so that P_{ext} and $\partial_\mu v_\mu$ can be neglected, (4.25) and (4.26) lead to the simplification

$$S_b = -\int d^4x\ \psi_b^\dagger \left(\frac{1}{2m_0}\partial_\mu\partial_\mu + i v_\mu^\alpha \sigma^\alpha \partial_\mu \right) \psi_b\ . \tag{4.34}$$

In the remainder of the paper it will also be assumed that

$$p_\mu \ll m_0 v_\mu^\alpha \tag{4.35}$$

for states with momenta p_μ corresponding to energies ~ 1 TeV or less (as would be the case if we had, e.g., $m_0 = a_0^{-1} \sim 10^{15}$ TeV and $v_\alpha^\mu \sim 1$), so that (4.34) reduces to just

$$S_b = \int d^4x \; \psi_b^\dagger i e_\alpha^\mu \sigma^\alpha \partial_\mu \psi_b \quad , \quad e_\alpha^\mu = -v_\mu^\alpha \; . \tag{4.36}$$

Up to this point there has been no distinction between contravariant and covariant vectors, and v_μ^α could have been written, e.g., v_α^μ, but it is important beyond this point that e_α^μ be treated as a contravariant 4-vector.

The above arguments also hold for fermions, with

$$S_f = \int d^Dx \left(-\frac{1}{2m_0} \Psi_f^\dagger \partial_M \partial_M \Psi_f - \mu_0 \Psi_f^\dagger \Psi_f + V_0 \Psi_f^\dagger \Psi_f \right) \tag{4.37}$$

$$\Psi_f \left(x^\mu, x^m\right) = \widetilde{\psi}_f^r \left(x^\mu\right) \widetilde{\psi}_{int}^r \left(x^m\right) \tag{4.38}$$

leading to the final result

$$S_f = \int d^4x \; \psi_f^\dagger i e_\alpha^\mu \sigma^\alpha \partial_\mu \psi_f \; . \tag{4.39}$$

The present theory thus yields the standard action for Weyl fermions, with the gravitational vierbein e_α^μ interpreted as essentially the "superfluid velocity" associated with the condensate Ψ_0. The path integral still has a Euclidean form, and the action for bosons is also not yet in standard form, but we will return to these points below.

V. GAUGE FIELDS

Let us now relax assumption (4.16) and allow U_{int} to vary with the external coordinates x^μ. Equation (4.10) is satisfied if (4.17) is generalized to

$$\left(-\frac{1}{2m_0} \partial_\mu \partial_\mu - \mu_{ext} \right) \Psi_{ext} \left(x^\mu\right) \Psi_{int} \left(x^m, x^\mu\right) = 0 \tag{5.1}$$

with Ψ_{int} required to satisfy the internal equation of motion (at each x^μ)

$$\left[\sum_m \frac{1}{2m_0} \left(-\frac{\partial^2}{\partial \left(x^m\right)^2} + \frac{a_0^2}{16} \frac{\partial^4}{\partial \left(x^m\right)^4} \right) + V_0 \left(x^m\right) - \mu_{int} \right] \Psi_{int} \left(x^m, x^\mu\right) = 0 \; . \tag{5.2}$$

The higher-derivative term of (3.14) has been retained and two integrations by parts have been performed. (In order to simplify the notation, we do not explicitly show the weak parametric dependence of μ_{int}, V_0, and n_{int} on x^μ.) This is a nonlinear equation because (at each x^μ) $V_0 \left(x^m\right)$ is mainly determined by $n_{int} = \Psi_{int}^\dagger \Psi_{int}$.

The internal basis functions satisfy the more general version of (4.28) with $\varepsilon_r = 0$:

$$\left[\sum_m \frac{1}{2m_0} \left(-\frac{\partial^2}{\partial (x^m)^2} + \frac{a_0^2}{16} \frac{\partial^4}{\partial (x^m)^4} \right) + V_0 (x^m) - \mu_{int} \right] \widetilde{\psi}^r_{int} (x^m, x^\mu) = 0 \; . \tag{5.3}$$

This is a linear equation because $V_0 (x^m)$ is now regarded as a known function.

The full path integral involving (3.14) contains all configurations of the fields, including those with nontrivial topologies. In the present theory, the geography of our universe includes a topological defect in the d-dimensional internal space which is analogous to a vortex. (See Appendix A.) The standard features of four-dimensional physics arise from the presence of this internal topological defect. For example, it compels the initial gauge symmetry to be $SO(d)$.

The behavior of the condensate and basis functions in the internal space is discussed in Appendices A and B. In (A15), the parameters $\overline{\phi}_i$ specify a rotation of $\Psi_{int} (x^m, x^\mu)$ as the external coordinates x^μ are varied, and according to (A16) the $\overline{J}_i$ satisfy the $SO(d)$ algebra

$$\overline{J}_i \overline{J}_j - \overline{J}_j \overline{J}_i = i c_{ij}^k \overline{J}_k \; . \tag{5.4}$$

For simplicity of notation, let

$$\langle r \,| Q | r' \rangle = \int d^d x \, \widetilde{\psi}^{r\dagger}_{int} Q \, \widetilde{\psi}^{r'}_{int} \quad \text{with} \quad \langle r \,| r' \rangle = \delta_{rr'} \tag{5.5}$$

for any operator Q, and in particular let

$$t_i^{rr'} = \left\langle r \,|\, \overline{J}_i | r' \right\rangle \tag{5.6}$$

with the matrices $t_i^{rr'}$ (which are constant according to (A17)) inheriting the $SO(d)$ algebra:

$$\begin{aligned} \left(t_i t_j - t_j t_i \right)^{rr'} &= \sum_{r''} \left\langle r \,|\, \overline{J}_i | r'' \right\rangle \left\langle r'' |\, \overline{J}_j | r' \right\rangle - \sum_{r''} \left\langle r \,|\, \overline{J}_j | r'' \right\rangle \left\langle r'' |\, \overline{J}_i | r' \right\rangle & (5.7) \\ &= \left\langle r \,|\, \overline{J}_i \, \overline{J}_j |\, r' \right\rangle - \left\langle r \,|\, \overline{J}_j \, \overline{J}_i |\, r' \right\rangle & (5.8) \\ &= i c_{ij}^k t_k^{rr'} \; . & (5.9) \end{aligned}$$

The t_i are the generators in the N_g-dimensional reducible representation determined by the physically significant solutions to (5.3), which spans all the irreducible (physical) gauge representations.

When $x^\mu \to x^\mu + \delta x^\mu$, Ψ_{int} and $\widetilde{\psi}^r_{int}$ rotate together, and (A15) implies that

$$\begin{aligned} \partial_\mu \widetilde{\psi}^r_{int} (x^m, x^\mu) &= \frac{\partial \overline{\phi}_i}{\partial x^\mu} \frac{\partial}{\partial \overline{\phi}_i} \widetilde{\psi}^r_{int} (x^m, x^\mu) & (5.10) \\ &= -i \, A^i_\mu \overline{J}_i \, \widetilde{\psi}^r_{int} (x^m, x^\mu) & (5.11) \end{aligned}$$

where

$$A_\mu^i = \frac{\partial \overline{\phi}_i}{\partial x^\mu} \; . \tag{5.12}$$

The A_μ^i will be interpreted below as gauge potentials. In other words, the gauge potentials are simply the rates at which the internal order parameter $\Psi_{int}\left(x^m, x^\mu\right)$ is rotating as a function of the external coordinates x^μ.

Let us return to the fermionic action (4.37). If (4.38) is written in the more general form

$$\Psi_f\left(x^\mu, x^m\right) = \widetilde{\psi}_f^r\left(x^\mu\right) \widetilde{\psi}_{int}^r\left(x^m, x^\mu\right) = U_{ext}\left(x^\mu\right) \psi_f^r\left(x^\mu\right) \widetilde{\psi}_{int}^r\left(x^m, x^\mu\right) \tag{5.13}$$

we have

$$\partial_\mu \Psi_f = U_{ext}\left(x^\mu\right) \left(\partial'_\mu - i m_0 e_\alpha^\mu \sigma^\alpha - i\, A_\mu^i \overline{J}_i\right) \psi_f^r \, \widetilde{\psi}_{int}^r \tag{5.14}$$

where the prime indicates that ∂'_μ does not operate on $\widetilde{\psi}_{int}^r$, and

$$\begin{aligned}
&\int d^d x\, \Psi_f^\dagger\, \partial_\mu \partial_\mu\, \Psi_f \\
&= \int d^d x\, \widetilde{\psi}_{int}^{r\dagger} \psi_f^{r\dagger} \left(\partial'_\mu - i m_0 e_\alpha^\mu \sigma^\alpha - i\, A_\mu^i \overline{J}_i\right) \left(\partial'_\mu - i m_0 e_{\alpha'}^\mu \sigma^{\alpha'} - i\, A_\mu^{i'} \overline{J}_{i'}\right) \psi_f^{r'} \widetilde{\psi}_{int}^{r'} && (5.15) \\
&= \psi_f^{r\dagger} \langle r| \left(\partial'_\mu - i m_0 e_\alpha^\mu \sigma^\alpha - i\, A_\mu^i \overline{J}_i\right) \sum_{r''} |r''\rangle\langle r''| \left(\partial'_\mu - i m_0 e_{\alpha'}^\mu \sigma^{\alpha'} - i\, A_\mu^{i'} \overline{J}_{i'}\right) |r'\rangle\, \psi_f^{r'} && (5.16) \\
&= \psi_f^{r\dagger} \left[\delta_{rr''}\left(\partial_\mu - i m_0 e_\alpha^\mu \sigma^\alpha\right) - i A_\mu^i t_i^{rr''}\right] \left[\delta_{r''r'}\left(\partial_\mu - i m_0 e_{\alpha'}^\mu \sigma^{\alpha'}\right) - i A_\mu^{i'} t_{i'}^{r''r'}\right] \psi_f^{r'} && (5.17) \\
&= \psi_f^\dagger \left[\left(\partial_\mu - i A_\mu^i t_i\right) - i m_0 e_\alpha^\mu \sigma^\alpha\right] \left[\left(\partial_\mu - i A_\mu^{i'} t_{i'}\right) - i m_0 e_{\alpha'}^\mu \sigma^{\alpha'}\right] \psi_f \; . && (5.18)
\end{aligned}$$

Then (4.37) becomes

$$S_f = \int d^4 x\, \psi_f^\dagger \left(-\frac{1}{2m_0} D_\mu D_\mu + \frac{1}{2} i e_\alpha^\mu \sigma^\alpha D_\mu + \frac{1}{2} D_\mu i e_\alpha^\mu \sigma^\alpha + \frac{1}{2} m_0 e_\alpha^\mu \sigma^\alpha e_{\alpha'}^\mu \sigma^{\alpha'} - \mu_{ext}\right) \psi_f$$

where

$$D_\mu = \partial_\mu - i A_\mu^i t_i \; . \tag{5.19}$$

With (4.25) and the approximations above (4.34), (4.26) implies that

$$S_f = \int d^4 x\, \psi_f^\dagger \left(-\frac{1}{2m_0} D_\mu D_\mu + i e_\alpha^\mu \sigma^\alpha D_\mu\right) \psi_f \; . \tag{5.20}$$

This is the generalization of (4.34) or (4.39) when the internal order parameter is permitted to vary as a function of the external coordinates x^μ. Again, for momenta and gauge potentials that are small compared to $m_0 e_\alpha^\mu$ (as would be the case at normal energies and fields if m_0

were comparable to a Planck energy and e_{α}^{μ} were ~ 1), the first term may be neglected. Furthermore, the entire treatment above can be repeated for the bosonic action, finally giving

$$S_f = \int d^4x \, \psi_f^{\dagger} i e_{\alpha}^{\mu} \sigma^{\alpha} D_{\mu} \psi_f \;\; , \;\; S_b = \int d^4x \, \psi_b^{\dagger} i e_{\alpha}^{\mu} \sigma^{\alpha} D_{\mu} \psi_b \; . \tag{5.21}$$

VI. TRANSFORMATION TO LORENTZIAN PATH INTEGRAL: FERMIONS

All of the foregoing is within a Euclidean picture, but we will now show that, in the case of fermions, there is a relatively trivial transformation to the more familiar Lorentzian description. A key point is that the low-energy *operator* $i e_{\alpha}^{\mu} \sigma^{\alpha} D_{\mu}$ in S_f is automatically in the correct Lorentzian form, even though the initial *path integral* is in Euclidean form. It is this fact which permits the following transformation to a Lorentzian path integral. Within the present theory, neither the fields nor the operators (nor the meaning of the time coordinate) need to be modified in performing this transformation.

In a locally inertial coordinate system, the Hermitian operator within S_f can be diagonalized to give

$$\begin{aligned} S_f &= \int d^4x \; \psi_f^{\dagger}(x) \; i\sigma^{\mu} D_{\mu} \, \psi_f(x) & (6.1) \\ &= \sum\nolimits_s \; \overline{\psi}_f^{\,*}(s) \; a(s) \; \overline{\psi}_f(s) & (6.2) \end{aligned}$$

where

$$\psi_f(x) = \sum_s U(x,s) \; \overline{\psi}_f(s) \quad , \quad \overline{\psi}_f(s) = \int d^4x \, U^{\dagger}(x,s) \; \psi_f(x) \tag{6.3}$$

with

$$\begin{aligned} i\sigma^{\mu} D_{\mu} U(x,s) &= a(s) \, U(x,s) & (6.4) \\ \int d^4x \, U^{\dagger}(x,s) \, U(x,s') &= \delta_{ss'} \quad , \quad \sum_s U(x,s) \, U^{\dagger}(x',s) = \delta(x-x') \; . & (6.5) \end{aligned}$$

Here, and in the following, x represents a point in external spacetime, and $U(x,s)$ is a multicomponent eigenfunction. There is an implicit inner product in

$$U^{\dagger}(x,s) \; \psi_f(x) \; = \; \sum_r U_r^{\dagger}(x,s) \; \psi_{f,r}(x) \tag{6.6}$$

with the $2N_g$ components of $\psi_f(x)$ labeled by $r = 1, ..., N_g$ (spanning all components of all irreducible gauge representations) and $a = 1, 2$ (labeling the components of Weyl spinors), and with s and (x, r, a) each having N values.

Evaluation of the present Euclidean path integral (a Gaussian integral with Grassmann variables) is then trivial for fermions; as usual,

$$Z_f = \int \mathcal{D}\,\psi_f^\dagger(x)\;\mathcal{D}\,\psi_f(x)\;\; e^{-S_f} \tag{6.7}$$

$$= \prod_{x,ra} \int d\,\psi_{f,ra}^*(x) \int d\,\psi_{f,ra}(x)\; e^{-S_f} \tag{6.8}$$

$$= \prod_s z_f(s) \tag{6.9}$$

with

$$z_f(s) = \int d\,\overline{\psi}_f^*(s) \int d\,\overline{\psi}_f(s)\; e^{-\overline{\psi}_f^*(s)\,a(s)\,\overline{\psi}_f(s)} \tag{6.10}$$

$$= a(s) \tag{6.11}$$

since the transformation is unitary [34]. Now let

$$Z_f^L = \int \mathcal{D}\,\overline{\psi}_f^\dagger(s)\;\mathcal{D}\,\overline{\psi}_f(s)\;\; e^{iS_f} \tag{6.12}$$

$$= \prod_s z_f^L(s) \tag{6.13}$$

where

$$z_f^L(s) = \int d\,\overline{\psi}_f^*(s) \int d\,\overline{\psi}_f(s)\; e^{i\,\overline{\psi}_f^*(s)\,a(s)\,\overline{\psi}_f(s)} \tag{6.14}$$

$$= -ia(s) \tag{6.15}$$

so that

$$Z_f^L = c_f Z_f \quad , \quad c_f = \prod_s (-i) \;\; . \tag{6.16}$$

This result holds for the path integral over an arbitrary time interval, with the fields, operator, and meaning of time left unchanged.

The transition amplitude from an initial state to a final state is equal to the path integral between these states, so transition probabilities are the same in the Lorentzian and Euclidean descriptions. This result is consistent with the fact that the classical equations of motion are also the same, since they follow from extremalization of the same action. Furthermore, using the method on pp. 290-291 or 302-303 of Ref. [34], it is easy to show that the magnitude $|G(x,x')|$ of the 2-point function is again the same, so particles propagate the same way in both descriptions. This result is also obtained in Appendix C with a different method.

When the inverse transformation from $\overline{\psi}_f$ to ψ_f is performed, we obtain

$$Z_f^L = \int \mathcal{D}\, \psi_f^\dagger\,(x)\ \mathcal{D}\, \psi_f\,(x)\ e^{iS_f} \tag{6.17}$$

with S_f having its form (6.1) in the coordinate representation.

One may perform calculations in either the path-integral formulation or the equivalent canonical formulation, which can now be obtained in the standard way: Let us use the notation $\int_a^b$ to indicate that the fields in a path integral are specified to begin in a state $|a\rangle$ at time t_a and end in state $|b\rangle$ at time t_b, and also to indicate that a path integral showing these limits has its conventional definition (so that it may differ by a normalization constant from Z_f^L as defined above). Then the Hamiltonian H_f is defined by

$$\langle b|\ U_f\,(t_b, t_a)\,|a\rangle\ =\ \int_a^b \mathcal{D}\, \psi_f^\dagger\,(x)\ \mathcal{D}\, \psi_f\,(x)\ e^{iS_f} \tag{6.18}$$

$$i\frac{d}{dt}U_f\,(t, t_a)\ =\ H_f\,(t)\, U_f\,(t, t_a) \quad , \quad U_f\,(t_a, t_a) = 1 \tag{6.19}$$

as in (9.14) of Ref. [34]. I.e., the time evolution operator $U_f\,(t_b, t_a)$ is defined to have the same effect as the path integral over intermediate states, and it is then straightforward to reverse the usual logic which leads from canonical quantization to path-integral quantization [34, 35].

VII. TRANSFORMATION TO STANDARD FIELDS AND LORENTZIAN PATH INTEGRAL: BOSONS

For bosons we can again perform the transformation (6.3) to obtain

$$S_b = \sum\nolimits_s \overline{\psi}_b^*\,(s)\ a\,(s)\ \overline{\psi}_b\,(s)\ \ . \tag{7.1}$$

The Euclidean path integral is

$$\begin{aligned} Z_b\ &=\ \int \mathcal{D}\, \psi_b^\dagger\,(x)\ \mathcal{D}\, \psi_b\,(x)\ \ e^{-S_b} &(7.2)\\ &=\ \prod_{x,ra} \int_{-\infty}^{\infty} d\ (\mathrm{Re}\, \psi_{b,ra}\,(x))\ \int_{-\infty}^{\infty} d\ (\mathrm{Im}\, \psi_{b,ra}\,(x))\ e^{-S_b} &(7.3)\\ &=\ \prod_s z_b\,(s) &(7.4) \end{aligned}$$

with

$$z_b\,(s) = \int_{-\infty}^{\infty} d(\mathrm{Re}\, \overline{\psi}_b\,(s)) \int_{-\infty}^{\infty} d(\mathrm{Im}\, \overline{\psi}_b\,(s))\, e^{-S_b}\ . \tag{7.5}$$

We will now show how this action can be put into a form which corresponds to scalar bosonic fields plus their auxiliary fields. First, if the gauge potentials A_μ^i were zero, we would have

$$i\sigma^\mu \partial_\mu U^0 (x, s) = a_0 (s) U^0 (x, s) \ . \tag{7.6}$$

Then

$$U^0 (x, s) = \mathcal{V}^{-1/2} u (s) e^{ip_s \cdot x} \quad , \quad p_s \cdot x = \eta_{\mu\nu} p_s^\mu x^\nu \quad , \quad \eta_{\mu\nu} = \mathrm{diag} (-1, 1, 1, 1) \tag{7.7}$$

(with $\mathcal{V}$ a four-dimensional normalization volume) gives

$$- \eta_{\mu\nu} \sigma^\mu p_s^\nu U^0 (x, s) = a_0 (s) U^0 (x, s) \tag{7.8}$$

where σ^μ implicitly multiplies the identity matrix for the multicomponent function $U^0 (x, s)$. A given 2-component spinor $u_r (s)$ has two eigenstates of $p_s^k \sigma^k$:

$$p_s^k \sigma^k u_r^+ (s) = |\overrightarrow{p}_s| \, u_r^+ (s) \quad , \quad p_s^k \sigma^k u_r^- (s) = - |\overrightarrow{p}_s| \, u_r^- (s) \tag{7.9}$$

where $\overrightarrow{p}_s$ is the 3-momentum, with magnitude $|\overrightarrow{p}_s|$. The multicomponent eigenstates of $i\sigma^\mu \partial_\mu$ and their eigenvalues $a_0 (s) = p_s^0 \mp |\overrightarrow{p}_s|$ thus come in pairs, corresponding to opposite helicities.

For nonzero A_μ^i, the eigenvalues $a (s)$ will also come in pairs, with one growing out of $a_0 (s)$ and the other out of its partner $a_0 (s')$ as the A_μ^i are turned on. To see this, first write (6.4) as

$$\left(i\partial_0 + A_0^i t_i \right) U (x, s) + \sigma^k \left(i\partial_k + A_k^i t_i \right) U (x, s) = a (s) U (x, s) \tag{7.10}$$

or

$$\left(i\partial_0 \delta_{rr'} + A_0^i t_i^{rr'} \right) U_{r'} (x, s) + P_{rr'} U_{r'} (x, s) - a (s) \delta_{rr'} U_{r'} (x, s) = 0 \tag{7.11}$$

$$P_{rr'} \equiv \sigma^k \left(i\partial_k \delta_{rr'} + A_k^i t_i^{rr'} \right) \tag{7.12}$$

with the usual implied summations over repeated indices. At fixed r, r' (and x, s), apply a matrix s which will diagonalize the 2×2 matrix $P_{rr'}$, bringing it into the form $p_{rr'} \sigma^3 + \overline{p}_{rr'} \sigma^0$, where $p_{rr'}$ and $\overline{p}_{rr'}$ are 1-component operators, while at the same time rotating the 2-component spinor $U_{r'}$:

$$s P_{rr'} s^{-1} = P'_{rr'} = p_{rr'} \sigma^3 + \overline{p}_{rr'} \sigma^0 \quad , \quad U'_{r'} = s U_{r'} \tag{7.13}$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{7.14}$$

But $P_{rr'}$ is traceless, and the trace is invariant under a similarity transformation, so $\overline{p}_{rr'} = 0$. Then the second term in (7.11) (for fixed r and r') becomes $s^{-1} p_{rr'} \sigma^3 U'_{r'}(x,s)$. The two independent choices

$$U'_{r'}(x,s) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \sigma^3 U'_{r'}(x,s) = U'_{r'}(x,s) \tag{7.15}$$

$$U'_{r'}(x,s) \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \sigma^3 U'_{r'}(x,s) = -U'_{r'}(x,s) \tag{7.16}$$

give $\pm s^{-1} p_{rr'} U'_{r'}(x,s)$. Now use $s^{-1} U'_{r'} = U_{r'}$ to obtain for (7.11)

$$\left(i\partial_0 \delta_{rr'} + A_0^i t_i^{rr'} \right) U_{r'}(x,s) \pm p_{rr'} U_{r'}(x,s) - a(s)\, \delta_{rr'} U_{r'}(x,s) = 0 \tag{7.17}$$

so (7.10) reduces to two sets of equations with different eigenvalues $a(s)$ and $a(s')$:

$$a(s) = a_1(s) + a_2(s) \quad , \quad a(s') = a_1(s) - a_2(s) \tag{7.18}$$

where these equations define $a_1(s)$ and $a_2(s)$. Notice that letting $\sigma^k \to -\sigma^k$ in (7.10) reverses the signs in (7.17), and results in $a(s) \to a(s')$:

$$\left(i\partial_0 + A_0^i t_i \right) U(x,s) - \sigma^k \left(i\partial_k + A_k^i t_i \right) U(x,s) = a(s')\, U(x,s) \ . \tag{7.19}$$

The action for a single eigenvalue $a(s)$ and its partner $a(s')$ is

$$\widetilde{s}_b(s) = \overline{\psi}_b^*(s)\ a(s)\ \overline{\psi}_b(s) + \overline{\psi}_b^*(s')\ a(s')\ \overline{\psi}_b(s') \tag{7.20}$$

$$= \overline{\psi}_b^*(s)\left(a_1(s) + a_2(s)\right)\overline{\psi}_b(s) + \overline{\psi}_b^*(s')\left(a_1(s) - a_2(s)\right)\overline{\psi}_b(s') \ . \tag{7.21}$$

For $a_1(s) > 0$, let us choose $a_2(s) > 0$ and define

$$\overline{\psi}_b(s') = a(s)^{1/2}\, \overline{\phi}_b(s) = \left(a_1(s) + a_2(s)\right)^{1/2} \overline{\phi}_b(s) \tag{7.22}$$

$$\overline{\psi}_b(s) = a(s)^{-1/2}\, \overline{F}_b(s) = \left(a_1(s) + a_2(s)\right)^{-1/2} \overline{F}_b(s) \tag{7.23}$$

so that

$$\widetilde{s}_b(s) = \overline{\phi}_b^*(s)\, \widetilde{a}(s)\, \overline{\phi}_b(s) + \overline{F}_b^*(s)\, \overline{F}_b(s) \quad , \quad a_1(s) > 0 \tag{7.24}$$

where

$$\widetilde{a}(s) = a(s)\, a(s') = a_1(s)^2 - a_2(s)^2 \ . \tag{7.25}$$

For $a_1(s) < 0$, let us choose $a_2(s) < 0$ and write

$$\overline{\psi}_b(s') = (-a(s))^{1/2}\,\overline{\phi}_b(s) = (-a_1(s) - a_2(s))^{1/2}\,\overline{\phi}_b(s) \tag{7.26}$$

$$\overline{\psi}_b(s) = (-a(s))^{-1/2}\,\overline{F}_b(s) = (-a_1(s) - a_2(s))^{-1/2}\,\overline{F}_b(s) \tag{7.27}$$

so that

$$\widetilde{s}_b(s) = -\left[\overline{\phi}_b^*(s)\,\widetilde{a}(s)\,\overline{\phi}_b(s) + \overline{F}_b^*(s)\,\overline{F}_b(s)\right] \quad , \quad a_1(s) < 0\ . \tag{7.28}$$

Then we have

$$S_b = \sum\nolimits_s' \widetilde{s}_b(s) \tag{7.29}$$

$$= \sum_{a_1(s)>0}' \left[\overline{\phi}_b^*(s)\,\widetilde{a}(s)\,\overline{\phi}_b(s) + \overline{F}_b^*(s)\,\overline{F}_b(s)\right] - \sum_{a_1(s)<0}' \left[\overline{\phi}_b^*(s)\,\widetilde{a}(s)\,\overline{\phi}_b(s) + \overline{F}_b^*(s)\,\overline{F}_b(s)\right] \tag{7.30}$$

where a prime on a summation or product over s means that only one member of an s, s' pair (as defined in (7.17) and (7.18)) is included.

Each of the transformations above from $\overline{\psi}_b$ to $\overline{\phi}_b$ and $\overline{F}_b$ has the form

$$\overline{\psi}_b(s') = A(s)^{1/2}\,\overline{\phi}_b(s) \quad , \quad \overline{\psi}_b(s) = A(s)^{-1/2}\,\overline{F}_b(s) \tag{7.31}$$

so that $d\overline{\psi}_b(s') = A(s)^{1/2}\,d\overline{\phi}_b(s)$, $d\overline{\psi}_b(s) = A(s)^{-1/2}\,d\overline{F}_b(s)$, and the Jacobian is $\prod_s' A(s)^{1/2}\,A(s)^{-1/2} = 1$. These transformations lead to the formal result

$$Z_b = \prod_{s\geq 0}' z_\phi(s) \cdot \prod_{s<0}' z_\phi(s) \cdot \prod_{a_1(s)\geq 0}' z_F(s) \cdot \prod_{a_1(s)<0}' z_F(s) \tag{7.32}$$

where

$$s < 0 \;\longleftrightarrow\; \widetilde{a}(s) = a_1(s)^2 - a_2(s)^2 < 0 \quad \text{if } a_1(s) \geq 0 \tag{7.33}$$

$$\longleftrightarrow\; \widetilde{a}(s) = a_1(s)^2 - a_2(s)^2 > 0 \quad \text{if } a_1(s) < 0 \tag{7.34}$$

with $s \geq 0$ otherwise, and where

$$z_\phi(s) = \int_{-\infty}^{\infty} d(\mathrm{Re}\,\overline{\phi}_b(s)) \int_{-\infty}^{\infty} d(\mathrm{Im}\,\overline{\phi}_b(s))\, e^{\mp|\widetilde{a}(s)|\left[\left(\mathrm{Re}\,\overline{\phi}_b(s)\right)^2 + \left(\mathrm{Im}\,\overline{\phi}_b(s)\right)^2\right]} \tag{7.35}$$

$$z_F(s) = \int_{-\infty}^{\infty} d(\mathrm{Re}\,\overline{F}_b(s)) \int_{-\infty}^{\infty} d(\mathrm{Im}\,\overline{F}_b(s))\, e^{\mp\left[\left(\mathrm{Re}\,\overline{F}_b(s)\right)^2 + \left(\mathrm{Im}\,\overline{F}_b(s)\right)^2\right]} \tag{7.36}$$

with the upper sign holding for $s \geq 0$ in (7.35) and $a_1(s) \geq 0$ in (7.36). (Also, $a_1(s) = 0$ is treated as a limiting case of $a_1(s) > 0$, and $\widetilde{a}(s) = 0$ requires the usual convergence factor in the path integral.)

The above formal expressions would be highly divergent for the modes with the lower signs. If $\overline{\Psi}_b'$ represents the field consisting of these modes, it therefore cannot be treated with the approximations used above. Its behavior is instead controlled by second-derivative and nonlinear terms with the same form as in (4.5), which respectively restrict the number of modes which have negative action and the contribution of each such mode. One might conjecture that $\left\langle \overline{\Psi}_b'^{\dagger} \overline{\Psi}_b' \right\rangle$ behaves effectively as a charge-neutral condensate density, very roughly analogous to the QCD vacuum condensate; but the only certainty is that treating the components of this field properly requires methods beyond those used in the present paper, so these modes will be omitted in the following discussion.

Then (7.32) is reduced to

$$Z_b = \prod_{s\geq 0}{}' z_\phi\left(s\right) \cdot \prod_{a_1(s)\geq 0}{}' z_F\left(s\right) \quad , \quad z_\phi\left(s\right) = \frac{\pi}{\left|\widetilde{a}\left(s\right)\right|} \quad , \quad z_F\left(s\right) = \pi \tag{7.37}$$

where $s \geq 0$ means

$$a_1\left(s\right) \geq \left|a_2\left(s\right)\right| \quad \text{if } a_1\left(s\right) \geq 0 \quad , \quad \left|a_1\left(s\right)\right| \leq \left|a_2\left(s\right)\right| \quad \text{if } a_1\left(s\right) < 0 \tag{7.38}$$

according to the definition below (7.32). Recall that if the gauge potentials A_μ^i were zero, we would have $a_1 = \omega$ and $a_2 = \left|\overrightarrow{p}\right|$, where ω is the frequency and $\overrightarrow{p}$ the 3-momentum.

Now let

$$Z_b^L = \int \mathcal{D}\,\overline{\phi}_b^{\dagger}\left(s\right) \mathcal{D}\,\overline{\phi}_b\left(s\right)\, \mathcal{D}\,\overline{F}_b^{\dagger}\left(s\right) \mathcal{D}\,\overline{F}_b\left(s\right)\; e^{iS_b} \tag{7.39}$$

$$= \prod_{s\geq 0}{}' z_\phi^L\left(s\right) \cdot \prod_{a_1(s)\geq 0}{}' z_F^L\left(s\right) \tag{7.40}$$

where

$$z_\phi^L\left(s\right) = \int_{-\infty}^{\infty} d(\mathrm{Re}\,\overline{\phi}_b\left(s\right)) \int_{-\infty}^{\infty} d(\mathrm{Im}\,\overline{\phi}_b\left(s\right)) e^{i\widetilde{a}(s)\left[\left(\mathrm{Re}\,\overline{\phi}_b(s)\right)^2 + \left(\mathrm{Im}\,\overline{\phi}_b(s)\right)^2\right]} \tag{7.41}$$

$$= i\frac{\pi}{\widetilde{a}\left(s\right)} \tag{7.42}$$

$$z_F^L\left(s\right) = \int_{-\infty}^{\infty} d(\mathrm{Re}\,\overline{F}_b\left(s\right)) \int_{-\infty}^{\infty} d(\mathrm{Im}\,\overline{F}_b\left(s\right)) e^{i\left[\left(\mathrm{Re}\,\overline{F}_b(s)\right)^2 + \left(\mathrm{Im}\,\overline{F}_b(s)\right)^2\right]} \tag{7.43}$$

$$= i\pi \tag{7.44}$$

since $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp\left(\, ia\left(x^2 + y^2\right)\right) = i\pi/a$. (Nuances of Lorentzian path integrals are discussed in, e.g., Ref. [34], p. 286.) We have then obtained

$$Z_b^L = c_b Z_b \tag{7.45}$$

where c_b is a product of factors of i.

To return to the coordinate representation, let

$$\Phi_b(x) = \sum_{s\geq 0}{}' U(x,s)\, \overline{\phi}_b(s) \quad , \quad \overline{\phi}_b(s) = \int d^4x\, U^\dagger(x,s)\, \Phi_b(x) \tag{7.46}$$

$$\mathcal{F}_b(x) = \sum_{a_1(s)\geq 0}{}' U(x,s)\, \overline{F}_b(s) \quad , \quad \overline{F}_b(s) = \int d^4x\, U^\dagger(x,s)\, \mathcal{F}_b(x)\ . \tag{7.47}$$

Recall that, according to (7.10) and (7.19),

$$i\sigma^\mu D_\mu U(x,s) = a(s)\, U(x,s) \quad , \quad i\overline{\sigma}^\mu D_\mu U(x,s) = a(s')\, U(x,s) \tag{7.48}$$

with $\overline{\sigma}^0 = \sigma^0$, $\overline{\sigma}^k = -\sigma^k$, $a(s) = a_1(s) + a_2(s)$, and $a(s') = a_1(s) - a_2(s)$, so the action in (7.39) becomes (with $\widetilde{a}(s) = a_1(s)^2 - a_2(s)^2$)

$$S_b = \sum_{s\geq 0}{}' \overline{\phi}_b^{\,*}(s)\, \widetilde{a}(s)\, \overline{\phi}_b(s) + \sum_{a_1(s)\geq 0}{}' \overline{F}_b^{\,*}(s)\, \overline{F}_b(s) \tag{7.49}$$

$$= \int d^4x \left[\frac{1}{2}\Phi_b^\dagger(x)\, i\overline{\sigma}^\mu D_\mu\, i\sigma^{\mu'} D_{\mu'}\Phi_b(x) + \frac{1}{2}\Phi_b^\dagger(x)\, i\sigma^\mu D_\mu\, i\overline{\sigma}^{\mu'} D_{\mu'}\Phi_b(x) + \mathcal{F}_b^\dagger(x)\, \mathcal{F}_b(x)\right] . \tag{7.50}$$

Now, just as an ordinary vector can be made to align with a single coordinate axis through a rotation, $\Phi_b(x)$ can be rotated (at each point x) to have only one component $\Phi'^1_{b,R}(x)$ in each irreducible gauge representation R, with $A^i_k \to A'^i_{R\,k}$ to keep the action invariant. (This is similar to a transformation to the unitary gauge of $SU(2)$.) Although $\Phi'^1_{b,R}(x)$ itself still has 2 components, the nonabelian term in (7.50) can now be diagonalized:

$$\begin{aligned}
\Phi'^{1\dagger}_{b,R}\sigma^k A'^i_{R\,k} t^{1r*}_i \sigma^{k'} A'^{i'}_{R\,k'} t^{r1}_{i'} \Phi'^1_{b,R} &= \frac{1}{2}\Phi'^{1\dagger}_{b,R}\left(\sigma^k\sigma^{k'} A'^i_{R\,k} t^{r1}_i A'^{i'}_{R\,k'} t^{r1}_{i'} + \sigma^{k'}\sigma^k A'^{i'}_{R\,k'} t^{r1}_{i'} A'^i_{R\,k} t^{r1}_i\right)\Phi'^1_{b,R} &(7.51)\\
&= \frac{1}{2}\Phi'^{1\dagger}_{b,R}\left(\sigma^k\sigma^{k'} + \sigma^{k'}\sigma^k\right) A'^i_{R\,k} t^{r1}_i A'^{i'}_{R\,k'} t^{r1}_{i'} \Phi'^1_{b,R} &(7.52)\\
&= \Phi'^{1\dagger}_{b,R} A'^i_{R\,k} t^{1r*}_i A'^{i'}_{R\,k} t^{r1}_{i'} \Phi'^1_{b,R}\ . &(7.53)
\end{aligned}$$

If we neglect $\partial_k\Phi_b(x)$, it is not difficult to show that the other terms can be similarly treated and the two components of Φ_b are decoupled. After an inverse rotation within (7.53) in each irreducible representation, (7.50) then becomes

$$S_b = \int d^4x \left[\Phi_b^\dagger(x)\, \eta^{\mu\nu} D_\mu D_\nu \Phi_b(x) + \mathcal{F}_b^\dagger(x)\, \mathcal{F}_b(x)\right] . \tag{7.54}$$

The two components of $\Phi_b(x)$ or $\mathcal{F}_b(x)$ are not independent, since they are both determined by $\overline{\phi}_b(s)$ or $\overline{F}_b(s)$.

The return to the coordinate representation is much simpler when $i\sigma^{\mu}D_{\mu}\, i\overline{\sigma}^{\mu'} D_{\mu'}$ and $\eta^{\mu\nu}D_{\mu}D_{\nu}$ have the same spectrum of eigenvalues, so that there are one-component orthonormal eigenfunctions $U_b\left(x,s\right)$ satisfying

$$\eta^{\mu\nu}D_{\mu}D_{\nu}U_b\left(x,s\right) = \widetilde{a}\left(s\right)U_b\left(x,s\right) \ . \tag{7.55}$$

In this case, let

$$\phi_b\left(x\right) = \sum_{s\geq 0}{}^{'} U_b\left(x,s\right)\overline{\phi}_b\left(s\right) \quad , \quad \overline{\phi}_b\left(s\right) = \int d^4x\, U_b^{\dagger}\left(x,s\right)\,\phi_b\left(x\right) \tag{7.56}$$

$$F_b\left(x\right) = \sum_{a_1(s)\geq 0}{}^{'} U_b\left(x,s\right)\overline{F}_b\left(s\right) \quad , \quad \overline{F}_b\left(s\right) = \int d^4x\, U_b^{\dagger}\left(x,s\right)\,F_b\left(x\right) \tag{7.57}$$

(with the number of values of x chosen to match the number of values of s for a unitary transformation), so that

$$S_b = \int d^4x\,\left[\phi_b^{\dagger}\left(x\right)\eta^{\mu\nu}D_{\mu}D_{\nu}\phi_b\left(x\right) + F_b^{\dagger}\left(x\right)F_b\left(x\right)\right] \tag{7.58}$$

with

$$Z_b^L = \int \mathcal{D}\,\phi_b^{\dagger}\left(x\right)\mathcal{D}\,\phi_b\left(x\right)\,\mathcal{D}\,F_b^{\dagger}\left(x\right)\mathcal{D}\,F_b\left(x\right)\; e^{iS_b} \ . \tag{7.59}$$

Again, this is the path integral for an arbitrary time interval, and one can define a time evolution operator and Hamiltonian as in Section VI. Notice, however, that the variations in $\phi_b\left(x\right)$ and $F_b\left(x\right)$ are constrained by the expansions of (7.56) and (7.57). These fundamental scalar bosonic fields should therefore exhibit nonstandard behavior. For example, as mentioned at the end of Appendix C, predictions based on radiative corrections will be altered.

VIII. SUPERSYMMETRY AND GRAVITY

The total action for fermions and bosons is

$$S_f + S_b = \int d^4x\,\left[\psi_f^{\dagger}\left(x\right)\,i\sigma^{\mu}D_{\mu}\,\psi_f\left(x\right) + \phi_b^{\dagger}\left(x\right)\eta^{\mu\nu}D_{\mu}D_{\nu}\phi_b\left(x\right) + F_b^{\dagger}\left(x\right)F_b\left(x\right)\right] \tag{8.1}$$

which in a general coordinate system becomes

$$S_f + S_b = \int d^4x\, e\,\left[\psi^{\dagger}\left(x\right)\,ie_{\alpha}^{\mu}\,\sigma^{\alpha}\widetilde{D}_{\mu}\,\psi\left(x\right) - g^{\mu\nu}\left(\widetilde{D}_{\mu}\phi\left(x\right)\right)^{\dagger}\widetilde{D}_{\nu}\phi\left(x\right) + F^{\dagger}\left(x\right)F\left(x\right)\right] \tag{8.2}$$

where $g_{\mu\nu}$ is the metric tensor and

$$e \;=\; \det e^{\alpha}_{\mu} = (-\det g_{\mu\nu})^{1/2} \;, \quad , \quad \widetilde{D}_{\mu} = D_{\mu} + e^{-1/2}\partial_{\mu}e^{1/2} \tag{8.3}$$

$$\psi\,(x) \;=\; e^{-1/2}\,\psi_f\,(x) \quad , \quad \phi\,(x) = e^{-1/2}\,\phi_b\,(x) \quad , \quad F\,(x) = e^{-1/2}F_b\,(x)\;. \tag{8.4}$$

We thus obtain the basic form for a supersymmetric action, where the fields ϕ, F, and ψ respectively consist of 1-component complex scalar bosonic fields, 1-component complex scalar auxiliary fields, and 2-component spin 1/2 fermionic fields. These fields span the various physical representations of the fundamental gauge group, which must be $SO(d)$ in the present theory, where $d = D - 4$. (More precisely, the group is $Spin(d)$, but $SO(d)$ is conventional terminology.) I.e., ψ includes the Standard Model fermions and Higgsinos, and ϕ corresponds to the sfermions and Higgses.

According to (8.2), the coupling of matter to gravity is very nearly the same as in standard general relativity. However, if the action is written in terms of the original fields ψ_f and ϕ_b, there is no factor of e. In other words, in the present theory the original action has the form $\int d^4x\,\overline{\mathcal{L}}$, whereas in standard physics it has the form $\int d^4x\,e\,\overline{\mathcal{L}}$. For an $\overline{\mathcal{L}}$ corresponding to a fixed vacuum energy density, there is then no coupling to gravity in the present theory, and the usual cosmological constant vanishes. This point was already made in an earlier paper [28], where the "cosmological constant" was defined to be the usual contribution to the stress-energy tensor from a constant vacuum Lagrangian density $\mathcal{L}_0$, which results from the factor of e. However, as was also pointed out in this 1996 paper, "There may be a much weaker term involving $\delta\mathcal{L}_0/\delta g^{\mu\nu}$, but this appears to be consistent with observation."

In the present theory, the Einstein-Hilbert action, with

$$\mathcal{L}_G = \frac{1}{2}\ell_P^{-2}\,e\;{}^{(4)}R\;, \tag{8.5}$$

and Maxwell-Yang-Mills action, with

$$\mathcal{L}_g = -\frac{1}{4}g_0^{-2}\,e\,F^i_{\mu\nu}F^i_{\rho\sigma}\,g^{\mu\rho}g^{\nu\sigma}\;, \tag{8.6}$$

must arise from the response of the vacuum. (Here g_0 is the coupling constant for the fundamental gauge group and $\ell_P^2 = 8\pi G$.) I.e., these terms are interpreted as essentially arising through a Sakharov "induced gravity" mechanism [36–41]. The gravitational and gauge curvatures must ultimately originate from 4-dimensional topological defects associated with the vierbein of (4.36) and gauge potentials of (5.12).

IX. CONCLUSION

Although a complete theory will require much additional work – including more detailed predictions for experiment – the following have been shown to arise as emergent properties from the initial statistical picture: Lorentz invariance, the general form of Standard-Model physics, an $SO(d)$ fundamental gauge theory (with e.g. $SO(10)$ permitting coupling constant unification and neutrino masses), supersymmetry, a gravitational metric with the form $(-,+,+,+)$, the correct coupling of matter fields to gravity, vanishing of the usual cosmological constant, and a mechanism for the origin of spacetime and quantum fields.

The new predictions of the present theory appear to be subtle, but include Lorentz violation at very high energies and nonstandard behavior of fundamental scalar bosons.

Appendix A: The internal space

The internal space of Section V is d-dimensional, with an $SO(d)$ (or more precisely $Spin(d)$) rotation group and its vector, spinor, etc. representations – for example, the **10** and **16** representations when $d = 10$. It may be helpful to begin with an analogy, however, in which external spacetime is replaced by the z-axis. The internal space is replaced by an xy-plane, with internal states described by 2-dimensional vector fields (rather than the higher-dimension vector and spinor fields considered below). One of these states is occupied by the condensate, and is represented by a vector $\boldsymbol{v}_1$ which points radially outward from the origin at all points in the xy-plane when $z = 0$. The other state is an additional basis function, represented by a vector $\boldsymbol{v}_2$ which is everywhere perpendicular to $\boldsymbol{v}_1$. But $\boldsymbol{v}_1$ is allowed to rotate as a function of z, so it has both radial and tangential components after a displacement along the z-axis. Then $\boldsymbol{v}_2$ is forced to rotate with $\boldsymbol{v}_1$ – i.e., the condensate – in order to preserve orthogonality.

Now let us turn to the actual internal space, first considering a set of d-dimensional vector fields $\widetilde{\psi}^r_{vec}$. Let $\widetilde{\psi}^0_{vec}$ represent the state occupied by a bosonic condensate. In the simplest picture, and at some fixed x_0^μ, only the rth component of the field $\widetilde{\psi}^r_{vec}$ is nonzero along some radial direction in the internal space, making the fields trivially orthogonal in that direction. Then, with x^μ still fixed, $\widetilde{\psi}^r_{vec}(x^m)$ in all other radial directions is obtained from the original $\widetilde{\psi}^r_{vec}(x_0^m)$ by rotating it to x^m. In other words, the field at each point in the

internal space is identical to the field that would be obtained at that point if the original field $\widetilde{\psi}^r_{vec}\left(x^m_0\right)$ were subjected to a rotation about the origin. This produces an isotropic configuration for the condensate and each basis function. As in (4.13) we can write

$$\widetilde{\psi}^r_{vec}\left(x^m\right) = U_{vec}\left(x^m, x^m_0\right) \widetilde{\psi}^r_{vec}\left(x^m_0\right) \ . \tag{A1}$$

Just as in the analogy, a field that is radial at x^m_0 will also be radial at all other points x^m. However, a general $\widetilde{\psi}^r_{vec}\left(x^m_0\right)$ permits a general vortex-like configuration of the condensate.

Also as in the analogy, the state $\widetilde{\psi}^0_{vec}$ of the condensate is allowed to rotate as a function of x^μ (because such a rotation does not alter the internal action). Since the other basis functions $\widetilde{\psi}^r_{vec}$ are required to remain orthogonal to $\widetilde{\psi}^0_{vec}$ and each other, they are required to rotate with the condensate. Then (A1) becomes more generally

$$\widetilde{\psi}^r_{vec}\left(x^m, x^\mu\right) = U_{vec}\left(x^m, x^m_0; x^\mu, x^\mu_0\right) \widetilde{\psi}^r_{vec}\left(x^m_0, x^\mu_0\right) \tag{A2}$$

with

$$\widetilde{\psi}^{r\,\dagger}_{vec}\left(x^m, x^\mu\right)\, \widetilde{\psi}^{r'}_{vec}\left(x^m, x^\mu\right) = \widetilde{\psi}^{r\,\dagger}_{vec}\left(x^m_0, x^\mu_0\right)\, \widetilde{\psi}^{r'}_{vec}\left(x^m_0, x^\mu_0\right) = \delta_{rr'} \tag{A3}$$

since

$$U^\dagger_{vec}\left(x^m, x^m_0; x^\mu, x^\mu_0\right) U_{vec}\left(x^m, x^m_0; x^\mu, x^\mu_0\right) = 1 \ . \tag{A4}$$

In general (with x^μ fixed), let $\widetilde{\psi}\left(\boldsymbol{x}\right)$ represent a multicomponent basis function with angular momentum j at a point $\boldsymbol{x}$ in the d-dimensional internal space. After a rotation about the origin specified by the $d \times d$ matrix $\boldsymbol{R}$, it is transformed to

$$\widetilde{\psi}'\left(\boldsymbol{x}\right) = \mathcal{R}\left(\boldsymbol{R}\right)\, \widetilde{\psi}\left(\boldsymbol{R}^{-1}\boldsymbol{x}\right) \tag{A5}$$

where $\mathcal{R}\left(\boldsymbol{R}\right)$ belongs to the appropriate representation of the group $Spin\left(d\right)$. However, we require that the field be isotropic, so that it is left unchanged after a rotation:

$$\widetilde{\psi}'\left(\boldsymbol{x}\right) = \widetilde{\psi}\left(\boldsymbol{x}\right) \ . \tag{A6}$$

Then we can define $\widetilde{\psi}\left(\boldsymbol{x}\right)$ at each value of the radial coordinate r by starting with a $\widetilde{\psi}\left(\boldsymbol{x}_0\right)$ and requiring that

$$\widetilde{\psi}\left(\boldsymbol{x}\right) = \mathcal{R}\left(\boldsymbol{R}\right)\, \widetilde{\psi}\left(\boldsymbol{x}_0\right) \quad , \quad \boldsymbol{x} = \boldsymbol{R}\,\boldsymbol{x}_0 \ . \tag{A7}$$

With this definition, $\widetilde{\psi}\left(\boldsymbol{x}\right)$ is a single-valued function of the coordinates only if j is an integer. If $j = 1/2$, e.g., $\widetilde{\psi}\left(\boldsymbol{x}\right)$ acquires a minus sign after a rotation of 2π, but it is single-valued on the $Spin\left(d\right)$ group manifold.

Multivalued functions are well-known in other similar contexts, such as the behavior of the phase of an ordinary superfluid order parameter $\psi_s = e^{i\theta_s} n_s^{1/2}$ around a vortex, which becomes discontinuous if it is required to be a single-valued function of the coordinates [41]. In the same way, $z^{1/2}$ exhibits a discontinuity across a branch cut if it is required to be a single-valued function and z is restricted to a single complex plane. I.e., $z^{1/2} = |z|^{1/2} e^{i\phi/2}$ gives $+|z|^{1/2}$ for $\phi = 0$ and $-|z|^{1/2}$ for $\phi = 2\pi$. But when defined on a pair of Riemann sheets, $z^{1/2}$ is a continuous function, and the same is true of $\widetilde{\psi}\left(\boldsymbol{x}\right)$ as we have defined it above, on the group manifold. The key idea in either case is to extend the manifold over which the function is defined, so that there are no artificial discontinuities.

A vectorial condensate and vectorial basis functions are appropriate for the simplest Higgs-like fields and their superpartners. Similarly, spinorial fields $\widetilde{\psi}^r_{sp}$ are appropriate for ordinary fermions, sfermions, and a possible primordial condensate occupying a state $\widetilde{\psi}^0_{sp}$. (In the present context, of course, "vector" and "spinor" refer only to properties in the internal space.) Again, let $\widetilde{\psi}^r_{sp}\left(x^m_0\right)$ represent a field along some radial direction in the internal space at some fixed x^μ_0. Then the field configuration for every point x^m is obtained by taking $\widetilde{\psi}^r_{sp}\left(x^m\right)$ to be identical to the field that would be obtained at that point if $\widetilde{\psi}^r_{sp}\left(x^m_0\right)$ were subjected to a rotation, with

$$\widetilde{\psi}^r_{sp}\left(x^m\right) = U_{sp}\left(x^m, x^m_0\right) \widetilde{\psi}^r_{sp}\left(x^m_0\right) \tag{A8}$$

as in (A7).

Again, the state $\widetilde{\psi}^0_{sp}$ of the condensate is allowed to rotate as a function of x^μ, and since the other basis functions $\widetilde{\psi}^r_{sp}$ must remain orthogonal to $\widetilde{\psi}^0_{sp}$ they are required to rotate with the condensate. The general version of (A8) is then

$$\widetilde{\psi}^r_{sp}\left(x^m, x^\mu\right) = U_{sp}\left(x^m, x^m_0; x^\mu, x^\mu_0\right) \widetilde{\psi}^r_{sp}\left(x^m_0, x^\mu_0\right) \ . \tag{A9}$$

The same reasoning applies to each irreducible representation, and thus to the combined set of fields $\widetilde{\psi}^r_{int}\left(x^m, x^\mu\right)$:

$$\widetilde{\psi}^r_{int}\left(x'^m, x'^\mu\right) = U_{int}\left(x'^m, x^m; x'^\mu, x^\mu\right) \widetilde{\psi}^r_{int}\left(x^m, x^\mu\right) \tag{A10}$$

with

$$\widetilde{\psi}^{r\,\dagger}_{int}\left(x'^{\,m},x'^{\,\mu}\right)\;\widetilde{\psi}^{r'}_{int}\left(x'^{\,m},x'^{\,\mu}\right)=\widetilde{\psi}^{r\,\dagger}_{int}\left(x^{m},x^{\mu}\right)\;\widetilde{\psi}^{r'}_{int}\left(x^{m},x^{\mu}\right)=\delta_{rr'}\;. \tag{A11}$$

So that the internal action will be unaffected as $x^{\mu}\rightarrow x'^{\,\mu}$, we require that the order parameter experience a uniform rotation, described by a matrix $\overline{\mathcal{R}}_{int}$ which is independent of x^m. Then U_{int} has the form

$$U_{int}\left(x'^{\,m},x^{m};x'^{\,\mu},x^{\mu}\right)=\overline{\mathcal{R}}_{int}\left(x'^{\,\mu},x^{\mu}\right)\;\mathcal{R}_{int}\left(x'^{\,m},x^{m}\right)\;. \tag{A12}$$

(Notice that (A12) is to be distinguished from a rotation about the origin, which is given by (A5), and which according to (A6) would leave $\widetilde{\psi}\left(x^{m}\right)$ unchanged rather than rotated at each point x^m.) It follows that

$$\widetilde{\psi}^{r}_{int}\left(x^{m},x^{\mu}\right)=\overline{\mathcal{R}}_{int}\left(x^{\mu},x^{\mu}_{0}\right)\;\widetilde{\psi}^{r}_{int}\left(x^{m},x^{\mu}_{0}\right)\;. \tag{A13}$$

We define the parameters $\delta\overline{\phi}_i$ by

$$\overline{\mathcal{R}}_{int}\left(x^{\mu}+\delta x^{\mu},x^{\mu}_{0}\right)=\overline{\mathcal{R}}_{int}\left(x^{\mu},x^{\mu}_{0}\right)\left(1-i\,\delta\overline{\phi}_{i}\,J_{i}\right) \tag{A14}$$

or

$$\delta\widetilde{\psi}^{r}_{int}\left(x^{m}\right)\;=\;-i\,\delta\overline{\phi}_{i}\,\overline{J}_{i}\,\widetilde{\psi}^{r}_{int}\left(x^{m}\right)\quad\text{as}\quad x^{\,\mu}\rightarrow x^{\,\mu}+\delta x^{\,\mu} \tag{A15}$$

$$\overline{J}_{i}\;=\;\overline{\mathcal{R}}_{int}\left(x^{\mu},x^{\mu}_{0}\right)J_{i}\,\overline{\mathcal{R}}^{-1}_{int}\left(x^{\mu},x^{\mu}_{0}\right) \tag{A16}$$

where the matrices J_i are the generators in the reducible representation of $Spin(d)$ corresponding to $\widetilde{\psi}^{r}_{int}$. The matrix elements of $\overline{J}_i$ are independent of x^μ:

$$\int d^{d}x\,\widetilde{\psi}^{r\dagger}_{int}\left(x^{m},x^{\mu}\right)\overline{J}_{i}\,\widetilde{\psi}^{r'}_{int}\left(x^{m},x^{\mu}\right)=\int d^{d}x\,\widetilde{\psi}^{r\dagger}_{int}\left(x^{m},x^{\mu}_{0}\right)J_{i}\,\widetilde{\psi}^{r'}_{int}\left(x^{m},x^{\mu}_{0}\right)\;. \tag{A17}$$

The primordial condensate is in a specific representation, but the basis functions in other representations are chosen to rotate with it according to (A13) and (A15).

It may be helpful to illustrate the above ideas by returning to the 2-dimensional analogy. Equation (A7) becomes

$$\boldsymbol{v}\left(\boldsymbol{x}\right)=\mathcal{R}_{vec}\,\boldsymbol{v}\left(\boldsymbol{x}_{0}\right)\;,\;\mathcal{R}_{vec}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\;,\;\boldsymbol{v}\left(\boldsymbol{x}_{0}\right)=\begin{pmatrix}R\left(r\right)\\ 0\end{pmatrix}\;\text{or}\;\begin{pmatrix}0\\ R\left(r\right)\end{pmatrix} \tag{A18}$$

for the vector representation and

$$s(\boldsymbol{x}) = \mathcal{R}_{sp}\, s(\boldsymbol{x}_0) \;\; , \;\; \mathcal{R}_{sp} = e^{-i\sigma_3\phi/2} \;\; , \;\; s(\boldsymbol{x}_0) = \begin{pmatrix} R(r) \\ 0 \end{pmatrix} \;\; \text{or} \;\; \begin{pmatrix} 0 \\ R(r) \end{pmatrix} \tag{A19}$$

for the spinor representation. The matrices corresponding to the J_i are

$$J_{vec} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad J_{sp} = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{A20}$$

Notice that ϕ_i is an angular coordinate in the internal space, whereas $\overline{\phi}_i$ is a parameter specifying the rotation of $\widetilde{\psi}^r_{int}$ at fixed x^m as x^μ is varied.

Appendix B: Example of solutions in internal space

Our goal in this appendix is merely to show that there are solutions with the form required in Appendix A, so we will look for solutions with the higher-derivative terms in (5.2) and (5.3) neglected, and with Ψ_{int} sufficiently small that $V_0(x^m)$ can also be neglected. Then (5.2) and (5.3) become

$$\left(-\frac{1}{2m_0}\partial_m\partial_m - \mu_{int}\right)\Psi_{int}(x^m, x^\mu) = 0 \quad , \quad \left(-\frac{1}{2m_0}\partial_m\partial_m - \mu_{int}\right)\widetilde{\psi}^r_{int}(x^m, x^\mu) = 0 \;. \tag{B1}$$

For simplicity of notation, let $\widetilde{\psi}^r_{int}(x^m, x^\mu)$ again be represented by $\widetilde{\psi}(\boldsymbol{x})$, with components $\widetilde{\psi}_p(\boldsymbol{x})$. Each component varies with position in the way specified by (A7) (together with the radial dependence of $\widetilde{\psi}(\boldsymbol{x}_0)$). It therefore has a kinetic energy given by $-(2m_0)^{-1}\partial_m\partial_m\widetilde{\psi}_p(\boldsymbol{x})$, and an orbital angular momentum given by the usual orbital angular momentum operators $\widehat{J}_i$ in d dimensions [42–46], which essentially measure how rapidly $\widetilde{\psi}_p(\boldsymbol{x})$ varies as a function of the angles ϕ_i.

The Laplacian $\partial_m\partial_m$ can be rewritten in terms of radial derivatives and the usual $\widehat{J}^2$, giving [42–44]

$$\left(-\frac{1}{r^{2K}}\frac{\partial}{\partial r}\left(r^{2K}\frac{\partial}{\partial r}\right) + \frac{\widehat{J}^2}{r^2} - 1\right)\widetilde{\psi}_p(\boldsymbol{x}) = 0 \quad , \quad K = \frac{d-1}{2} \tag{B2}$$

after rescaling of the radial coordinate r. In addition, it is shown in Refs. [42–44] that

$$\widehat{J}^2\widetilde{\psi}_p(\boldsymbol{x}) = j\,(j+d-2)\,\widetilde{\psi}_p(\boldsymbol{x}) \tag{B3}$$

where j is the orbital angular momentum quantum number, as defined on p. 677 of Ref. [43], but with this definition extended to half-integer values of m_α and j. Normally, of course, only integer values of these orbital quantum numbers are permitted [47]. However, the functions $\widetilde{\psi}_p(\boldsymbol{x})$ as defined in Appendix A can have $j = 1/2$ etc. (in which case they are multivalued functions of the coordinates but single-valued functions on the group manifold, as discussed below (A7)). Also, the demonstration of (B3) in Ref. [43] can be extended in the present context to half-integer j, because it employs raising and lowering operators. (At each $\boldsymbol{x}$, $\widetilde{\psi}_p$ is a linear combination of states with different values of m_α, but (B3) still holds.) For each $\widetilde{\psi}_p(\boldsymbol{x})$ the radial wavefunction then satisfies

$$\left[-\frac{1}{r^{2K}}\frac{d}{dr}\left(r^{2K}\frac{d}{dr}\right)+\frac{j\,(j+d-2)}{r^2}-1\right]R\,(r)=0\;. \tag{B4}$$

It may be helpful once again to consider the 2-dimensional analogy of Appendix A, where the orbital angular momentum operator is

$$\widehat{J}=-i\partial/\partial\phi\;. \tag{B5}$$

For the vector representation, (A18) implies that the kinetic energy is given by

$$\partial_m\partial_m\boldsymbol{v}\,(\boldsymbol{x}) = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\boldsymbol{v}\,(\boldsymbol{x}_0) \tag{B6}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\right]\boldsymbol{v}\,(\boldsymbol{x}) \tag{B7}$$

in agreement with (B4) for $j = 1$. For the spinor representation, (A19) gives

$$\partial_m\partial_m s\,(\boldsymbol{x}) = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]e^{-i\sigma_3\phi/2}\,s\,(\boldsymbol{x}_0) \tag{B8}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1/4}{r^2}\right]s\,(\boldsymbol{x}) \tag{B9}$$

in agreement with (B4) for $j = 1/2$.

Equation (B4) can be further reduced to [45, 46]

$$\left[-\frac{d^2}{dr^2}+\frac{k\,(k-1)}{r^2}-1\right]\chi\,(r)=0\quad,\quad k=j+K=j+\frac{d-1}{2} \tag{B10}$$

where $\chi\,(r) \equiv r^K R\,(r)$. It is then easy to show that

$$\chi\,(r)\propto r^k\quad\text{as } r\to 0\quad,\quad \chi\,(r)\propto\sin\,(r+\delta)\quad\text{as } r\to\infty \tag{B11}$$

where δ is a phase.

As a specific example, for $d = 10$ and $j = 5/2$ (i.e., the usual spinorial representation) we have

$$\left(-\frac{d^2}{dr^2} + \frac{42}{r^2} - 1 \right) \chi \left(r \right) = 0 \tag{B12}$$

with the exact solution

$$\chi \left(r \right) \;=\; c\, X^{1/2} \sin \left(\int \frac{dr}{X} \right) \tag{B13}$$

$$X \;=\; 1 + \frac{21}{r^2} + \frac{630}{r^4} + \frac{18\,900}{r^6} + \frac{496\,125}{r^8} + \frac{9\,823\,275}{r^{10}} + \frac{108\,056\,025}{r^{12}} \tag{B14}$$

(where c is a normalization constant), which reduces to

$$\chi \left(r \right) \propto r^7 \quad \text{as } r \to 0 \quad , \quad \chi \left(r \right) \propto \sin \left(r + \delta \right) \quad \text{as } r \to \infty \;. \tag{B15}$$

Again, these examples are merely meant to indicate that solutions with the right basic form do exist. More realistic solutions will require detailed numerical calculations.

Appendix C: Euclidean and Lorentzian Propagators

For Weyl fermions, the Euclidean 2-point function is

$$G_f \left(x_1, x_2 \right) = \left\langle \psi_f \left(x_1 \right) \psi_f^\dagger \left(x_2 \right) \right\rangle = \frac{\int \mathcal{D}\, \psi_f^\dagger\, \mathcal{D}\, \psi_f\, \psi_f \left(x_1 \right) \psi_f^\dagger \left(x_2 \right)\, e^{-S_f}}{\int \mathcal{D}\, \psi_f^\dagger\, \mathcal{D}\, \psi_f\, e^{-S_f}} \tag{C1}$$

$$= \frac{\prod_s \int d\overline{\psi}_f^{\,*} \left(s \right) \int d\overline{\psi}_f \left(s \right)\, e^{-\overline{\psi}_f^{\,*} (s) a(s) \overline{\psi}_f (s)} \sum_{s_1, s_2} \overline{\psi}_f \left(s_1 \right) \overline{\psi}_f^{\,*} \left(s_2 \right) U \left(x_1, s_1 \right)\, U^\dagger \left(x_2, s_2 \right)}{\prod_s \int d\overline{\psi}_f^{\,*} \left(s \right) \int d\overline{\psi}_f \left(s \right)\, e^{-\overline{\psi}_f^{\,*} (s) a(s) \overline{\psi}_f (s)}} \tag{C2}$$

where (6.2) and (6.3) have been used. In a term with $s_2 \neq s_1$, the numerator contains the factor

$$\int d\overline{\psi}_f^{\,*} \left(s_1 \right) \int d\overline{\psi}_f \left(s_1 \right)\, e^{-\overline{\psi}_f^{\,*} (s_1) a(s_1) \overline{\psi}_f (s_1)} \overline{\psi}_f \left(s_1 \right) = 0 \tag{C3}$$

according to the rules for Berezin integration. But a term with $s_2 = s_1$ contributes

$$\frac{\int d\overline{\psi}_f^{\,*} \left(s_1 \right) \int d\overline{\psi}_f \left(s_1 \right)\, e^{-\overline{\psi}_f^{\,*} (s_1) a(s_1) \overline{\psi}_f (s_1)} \overline{\psi}_f \left(s_1 \right) \overline{\psi}_f^{\,*} \left(s_1 \right)}{\int d\overline{\psi}_f^{\,*} \left(s_1 \right) \int d\overline{\psi}_f \left(s_1 \right)\, e^{-\overline{\psi}_f^{\,*} (s_1) a(s_1) \overline{\psi}_f (s_1)}} U \left(x_1, s_1 \right)\, U^\dagger \left(x_2, s_1 \right)$$

$$= a \left(s_1 \right)^{-1} U \left(x_1, s_1 \right)\, U^\dagger \left(x_2, s_1 \right) \tag{C4}$$

so

$$G_f\left(x_1,x_2\right)=\sum\nolimits_s \overline{G}_f\left(s\right)U\left(x_1,s\right)\,U^{\dagger}\left(x_2,s\right)\quad,\quad \overline{G}_f\left(s\right)=a\left(s\right)^{-1}\;. \tag{C5}$$

If the $U\left(x,s\right)$ used to represent $\psi_f\left(x\right)$ are a complete set, the propagator $G_f\left(x,x'\right)$ is a true Green's function:

$$L_f\left(x\right)U\left(x,s\right)=a\left(s\right)U\left(x,s\right)\quad,\quad \psi_f\left(x\right)=\sum_s U\left(x,s\right)\,\overline{\psi}_f\left(s\right) \tag{C6}$$

and $\sum_s U\left(x,s\right)\,U^{\dagger}\left(x',s\right)=\delta\left(x-x'\right)$ imply that

$$L_f\left(x\right)G_f\left(x,x'\right)=\delta\left(x-x'\right) \tag{C7}$$

as usual.

The treatment for scalar bosons is similar:

$$G_b\left(x_1,x_2\right)=\left\langle \phi_b\left(x_1\right)\phi_b^{\dagger}\left(x_2\right)\right\rangle=\frac{\int\mathcal{D}\,\phi_b^{\dagger}\;\mathcal{D}\,\phi_b\,\phi_b\left(x_1\right)\phi_b^{\dagger}\left(x_2\right)\;e^{-S_f}}{\int\mathcal{D}\,\phi_b^{\dagger}\,\mathcal{D}\,\phi_b\;e^{-S_f}} \tag{C8}$$

$$=\frac{\prod_s\int_{-\infty}^{\infty}d\,\mathrm{Re}\,\overline{\phi}_b\left(s\right)\int_{-\infty}^{\infty}d\,\mathrm{Im}\,\overline{\phi}_b\left(s\right)\;e^{-\widetilde{a}\left(s\right)\left[\left(\mathrm{Re}\,\overline{\phi}_b\left(s\right)\right)^2+\left(\mathrm{Im}\,\overline{\phi}_b\left(s\right)\right)^2\right]}\sum_{s_1,s_2}\overline{\phi}_b\left(s_1\right)\overline{\phi}_b^{\,*}\left(s_2\right)}{\prod_s\int_{-\infty}^{\infty}d\,\mathrm{Re}\,\overline{\phi}_b\left(s\right)\int_{-\infty}^{\infty}d\,\mathrm{Im}\,\overline{\phi}_b\left(s\right)\;e^{-\widetilde{a}\left(s\right)\left[\left(\mathrm{Re}\,\overline{\phi}_b\left(s\right)\right)^2+\left(\mathrm{Im}\,\overline{\phi}_b\left(s\right)\right)^2\right]}}$$
$$\times U_b\left(x_1,s_1\right)\,U_b^{\dagger}\left(x_2,s_2\right) \tag{C9}$$

where

$$L_b\left(x\right)U_b\left(x,s\right)=\widetilde{a}\left(s\right)U_b\left(x,s\right)\quad,\quad \phi_b\left(x\right)=\sum_s U_b\left(x,s\right)\,\overline{\phi}_b\left(s\right)\;. \tag{C10}$$

In a term with $s_2\neq s_1$, the numerator contains the factor

$$\int_{-\infty}^{\infty}d\,\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\int_{-\infty}^{\infty}d\,\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\;e^{-\widetilde{a}\left(s_1\right)\left[\left(\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\right)^2+\left(\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\right)^2\right]}\left[\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)+i\,\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\right]$$
$$=0 \tag{C11}$$

since the integrand is odd. But a term with $s_2=s_1$ contains the factor

$$\frac{\int_{-\infty}^{\infty}d\,\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)e^{-\widetilde{a}\left(s_1\right)\left(\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\right)^2}\left(\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\right)^2}{\int_{-\infty}^{\infty}d\,\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\;e^{-\widetilde{a}\left(s_1\right)\left(\mathrm{Re}\,\overline{\phi}_b\left(s_1\right)\right)^2}}+\frac{\int_{-\infty}^{\infty}d\,\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)e^{-\widetilde{a}\left(s_1\right)\left(\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\right)^2}\left(\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\right)^2}{\int_{-\infty}^{\infty}d\,\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\;e^{-\widetilde{a}\left(s_1\right)\left(\mathrm{Im}\,\overline{\phi}_b\left(s_1\right)\right)^2}}$$
$$=\widetilde{a}\left(s_1\right)^{-1} \tag{C12}$$

so

$$G_b\left(x_1,x_2\right)=\sum\nolimits_s \overline{G}_b\left(s\right)U_b\left(x_1,s\right)\,U_b^{\dagger}\left(x_2,s\right)\quad,\quad \overline{G}_b\left(s\right)=\widetilde{a}\left(s\right)^{-1}\;. \tag{C13}$$

As usual, $a\left(s\right)$ and $\widetilde{a}\left(s\right)$ contain a $+i\epsilon$ which is associated with a convergence factor in the path integral (and which gives a well-defined inverse).

The above are the propagators in the Euclidean formulation. The Lorentzian propagators are obtained through the same procedure with $a\left(s\right)\rightarrow -ia\left(s\right)$ and $\widetilde{a}\left(s\right)\rightarrow -i\widetilde{a}\left(s\right)$:

$$\overline{G}_f^L\left(s\right)=ia\left(s\right)^{-1} \quad , \quad \overline{G}_b^L\left(s\right)=i\widetilde{a}\left(s\right)^{-1} \ . \tag{C14}$$

The propagators in the Euclidean and Lorentzian formulations thus differ by only a factor of i. More generally, in the present picture, the action, fields, operators, classical equations of motion, quantum transition probabilities, propagation of particles, and meaning of time are the same in both formulations. (However, this equivalence holds only when there are no self-interactions of the form $\left(\phi_b^\dagger\phi_b\right)^2$, and (4.5) appears to imply that the original Euclidean formulation should be used when terms like these are significant.)

For a single noninteracting bosonic field with a mass m_b, the basis functions are

$$U_b\left(x,p\right)=\mathcal{V}^{-1/2}e^{ip\cdot x}=\mathcal{V}^{-1/2}e^{-i\omega t}e^{i\overrightarrow{p}\cdot\overrightarrow{x}} \tag{C15}$$

so with $s\rightarrow p$ we have

$$\widetilde{a}\left(p\right)=\omega^2-\left|\overrightarrow{p}\right|^2-m_b^2+i\epsilon \tag{C16}$$

and

$$\overline{G}_b\left(p\right) = \frac{1}{\omega^2-\left|\overrightarrow{p}\right|^2-m_b^2+i\epsilon} \tag{C17}$$

$$\overline{G}_b^L\left(p\right) = \frac{i}{\omega^2-\left|\overrightarrow{p}\right|^2-m_b^2+i\epsilon} \ . \tag{C18}$$

Notice that (C5) and (C13) hold even when the basis functions in (C6) or (C10) are not a complete set. The propagators for the scalar bosons of Section VII therefore have the form of (C13), but include only those modes with $s\geq 0$, as specified in (7.38). This implies that radiative corrections will be modified.

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